A FREQUENCY CONTROL FOR DIGITAL SWITCHING NETWORKS

by

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STATEMENT

The work presented in this Thesis is original, with the exceptions stated below, and has not been submitted for another degree of this or any other University. The exceptions are

1. The work contained in the candidate's M.Sc. dissertation (University of Warwick, June 1967) which is reproduced in part in sections 3.1.1, 6.3.1, 6.3.2 and as Theorem II in section 9.1. This work is repeated here for the sake of completeness.

2. The treatment of the stability of sampled single-ended systems with filters (section 3.3.3), the result for which was first obtained by Mr. P. C. Parks.

3. Of the analytical techniques used in this Thesis, the use of the Gershgorin Circle Theorem was originally suggested by Mr. Parks and the use of the Diagonal Dominance condition, which is closely related to the Gershgorin Theorem, was suggested by studying references 12 and 20.

4. The work on operation modes in section 5.2, which is different from, and independent of, that of Saito *et al.*, was stimulated by discussions on Lattice Theory with Professor E. C. Zeeman and Mr. Parks.

5. The work in section 6.3 on the estimation of transients by the use of Liapunov functions is based on the work of the Russian author N. G. Chetaev. The Liapunov function for the linear system (section 6.3.2) was suggested by Mr. Parks. The extension of this to non-linear systems in section 6.3.3 is, however, original.

M. R. [signature]
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ABSTRACT

Pulse code modulation is being introduced into telephone networks to increase the capacity of existing multi-pair cables. It is possible to use a switched digital communications network for data as well as for p.c.m. telephone traffic if all the clock oscillators in the network operate in phase at all times. Various control systems have been proposed to achieve the required clock synchronisation without the use of a central reference.

These control systems are based on the application of phase locking to a large number of interconnected oscillators. Some proposals are for non-linear systems, but this Thesis is largely restricted to the discussion of linear systems.

Two main types of linear system have been proposed. The first, known as the single-ended system, minimises the amount of buffer storage needed at the inputs to the exchanges. These stores allow for the variation of line delay with temperature, as well as accommodating small phase errors. The resulting network operating frequency is substantially dependent on the line delays.

The alternative is the double-ended system. This eliminates the interdependence of frequency and line delay, but involves a rather more complicated stability condition. Filters, which are an essential part of the system hardware, affect the stability of both systems. Their effects are examined in the Thesis.

The system equations derived in this Thesis include the line

delays, the effects of lumped filters and the starting conditions. They are obtained for both the single-and double-ended systems. A continuous model is assumed, but the systems are in fact sampled at a high rate. An alternative set of equations has been derived to allow the effects of the sampling rate to be investigated.

The behaviour of the systems is examined by means of the mathematics of multivariable control systems. This has allowed many important results to be deduced without exact knowledge of the connections between the exchanges, or even the number of exchanges.

The stability criteria in this Thesis are all sufficient, but not generally necessary. As a result, it has been possible to give the criteria in forms which may be checked at each exchange independently of the rest of the network. It is possible to add to the network without disturbing remote exchanges. The criteria are given in terms of the basic system parameters, such as gains and filter time constants. This Thesis includes a full treatment of the double-ended system, including some of the effects of unbalanced gains in the error paths. The results are more comprehensive than any others published to date, and include the effects of sampling on both classes of system.

The factors determining the final frequency of the synchronised network are examined. The double-ended system is treated in a general manner, which yields a formula for unbalanced systems as well as the more usual balanced system. An extreme case of this is the single-ended system. This general formula is expressed in terms of the basic system parameters, which are easily measured. It is shown that the frequency depends upon the variations of the line delays from integer multiples of one frame period, and not their absolute values.

The phase comparators used in the locked oscillators are by nature periodic; a piecewise-linear form is considered to be most suitable. The periodicity causes a number of stable states to exist with different phase differences. Some of these differences may cause the buffer store capacities to be insufficient. A new set of system equations, in terms of principle phase differences, is evolved. This leads to a phase-space representation of the system state, which has links with the method used to examine transient behaviour. Although it is not possible to give a general method for finding all the possible modes for a given network, a necessary condition for a 'wrong mode', with some large phase differences, can be found. This has suggested a technique for the detection of such modes and their release to the desired mode with substantially zero phase differences.

The investigation of transients and the effects of noise is not complete. There is, as yet, no quantitative analysis of the noise spectrum at the lower frequencies which will affect the control system. Some indications of the critical frequencies are given, to allow such measurements to be made. The investigation of transients shows that in most networks some phase differences will be increased before the network relaxes. A Liapunov estimate is used, but this is limited to systems without filters or line delays.

The results reported in this Thesis allow the system designer to determine the values of the controllable parameters. Some recommendations are given, and suggestions are made for the design of equipment to automatically set the network in the required mode of synchronised operation. A field trial of these systems is required, and proposals being studied by the British Post Office are based on results given in this Thesis. Such a trial should take place over lines carrying signals of the same statistical form as live telephone traffic. This will allow the deduction of further results concerning the transient response of the systems.

PREFACE

The work described in this Thesis was performed in the School of Engineering Science at The University of Warwick between October 1966 and September 1969. The study was undertaken on behalf of the British Post Office Research Department, and has been supported throughout by a Post Office Postgraduate Studentship awarded to me in 1966.

The first year was spent following the course of lectures leading to the award of the degree of Master of Science in automatic control. Part of the requirement for this degree was a project dissertation. Under the supervision of Mr. P. C. Parks, this took the form of a study of stability, final frequency and transient response of some of the control schemes described in this Thesis. Part of this work is reiterated here for the sake of completeness; the Statement of Originality indicates these parts.

The problem of p.c.m. network synchronisation is of considerable current interest, in view of the recent widespread introduction of p.c.m. telephone communications. This interest is world-wide and is illustrated by the number of papers published in the period of the study.

A number of these papers are closely related to the present study, but it is claimed that this Thesis contains the most comprehensive set of results so far obtained, certainly as far as linear continuous and sampled systems are concerned.

Work contained in this Thesis has already been reported in a number of published papers, which are contained in Appendix 9.3. The work on operating modes has led to two British Patent Applications (references 40 and 41).

The results contained in this Thesis have provided a basis for a proposed field-trial by the Post Office of a network of synchronised p.c.m. clock oscillators.

During the early part of the course of study, I was a member of a working party set up by the (P.O.-Industry) Joint Electronic Research Council to advise on aspects of network synchronisation. Certain lines of thought were suggested by the discussions of this working party, but the work described here is entirely original and is not derived from the work of other members of the working party.

ACKNOWLEDGEMENTS

I should like to thank Mr. P. C. Parks for his active supervision of this work and the staff of the Post Office Research Station, Dollis Hill, particularly Mr. W. T. Duerdoth, for most useful discussions and helpful information.

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1 INTRODUCTION

1.1 P.C.M., Digital Exchanges and Synchronised Clocks

Pulse Code Modulation (p.c.m.) was invented by A. H. Reeves in 1939[1], but it was not until the advent of the transistor that it became commercially useful. A single speech or other analogue signal is sampled at fixed intervals and the samples transmitted as a series of coded pulses which indicate the voltage level at the time of the sample. At the receiving terminal the code pulses are used to reform a series of samples which may then be reconstituted into a continuous signal in the usual way.

The coding process introduces amplitude quantisation and the sampling process limits the transmitted bandwidth to one half of the sampling frequency. There is thus a lower limit to the sampling frequency, which for speech is generally 8 kHz. Quantisation introduces noise as there is an error in the signal recovery. Choice of a suitable law ensures that the noise amplitude is a fixed proportion of the signal amplitude. In practice, this noise is less objectionable than the cross-talk noise encountered on ordinary analogue transmission paths. This latter form of noise is entirely removed from the signal by the digital nature of the signals, provided that digital repeaters are used at intervals.

The bandwidth needed on the line is always greater than that of the unmodulated analogue signal. To permit the use of two digital links for one call, with an intermediate analogue path, we need to use at least 7 bits to code each sample. In practice, a signalling channel is also required, and this introduces an eighth bit to each sample. We thus require a total bandwidth of 64 kHz for the 4 kHz speech channel.

The advantage of p.c.m. is that ordinary telephone lines of the type used to link exchanges (junction cables) can be made to pass a digital message stream of 1·536 Mb/S when digital repeaters are inserted at 2000 yard intervals. At these intervals, manholes are provided to give space for inductive loading of the cables, and digital repeaters can be built that occupy the same space as the loading coils, which are removed. Such a 1·536 Mb/S signal would result from the pulses of 24 channels of 4 kHz being interleaved in some way. Even allowing for the fact that two separate pairs of wire are needed to provide two directions of speech, there is a 12-fold increase in traffic capacity of the cables when p.c.m. is used. The cost of the terminal equipment is such that the digital system is more economic than separate pairs of wire for all new circuits over 12 miles length, at present prices.

The system used in Great Britain and most other countries is for 24 channels to be multiplexed by interleaving complete 'slots', or groups of eight bits, each group representing a single sample of some channel. Additional information replaces the signalling channels of certain 'frames' to identify each 'frame' of 192 bits, representing a single sample of all the channels.

In order to reduce the quantising noise on connections using more than one digital link, Vaughan has suggested[2] that the digital message should be switched without demodulation. This would make the transmission quality the same on all calls, regardless of total distance. In principle, it is necessary that any slot from one incoming p.c.m. system can be mixed with slots from other incoming systems to form a frame in an outgoing system. Two problems arise from this. The first, discussed by Walker and Duerdoth[3], is that for traffic reasons it must be possible to delay and advance slots relative to their proper timing so that they coincide with the gaps in the outgoing information. Advancement is achieved by delaying the information (24-x) slots so that it is apparently x slots earlier. Generally it is proposed that traffic be offered to an outgoing route in, first, the incoming time channel and then in progressively later channels until a free time slot is found. The return path information must, of course, be 'advanced' by the corresponding amount. It is thus possible to use the same delay element for both the x and (24-x) slot delays. The second problem is that the information must be presented to the switches and delay devices in the same phase relation. All outgoing information is transmitted under the control of the exchange 'clock' oscillator, and, if independent clocks are used at the various exchanges, some means must be found to operate them at a common frequency and in phase. This type of synchronisation produces a set of 'homochronous' clocks.

The phases will be displaced by the line propagation delays. Unless the line delays around any closed loop exchanges are arranged to be a multiple of the basic sampling period of 125 μS, this will introduce a need for further information storage within the exchange. Such a delay corresponds to about 17 miles of cable.

The line delays vary with temperature of the cables so some means of compensation is required. This can be done by small 'buffer stores' at the incoming end of each line, which would produce a complementary variation.

One form of such a store is a number of bistable devices which are addressed by counters locked to the frame-start signals in the message stream. Similar counters are used in the demodulators to assign the correct meaning to each bit of the message. When the received bit is of value 1 the bistable element is set, otherwise it is reset. The state of each store element in turn is read under the control of similar counters locked to the exchange clock.
These counters would in any case be needed to operate the switches. The time between writing and reading a given store element is the instantaneous value of store delay which will be a function of both the line delay and any true phase difference between the clocks used at the two exchanges.

It is thus possible to generate a signal which indicates the store 'fill', within the period of the counters used. Several control systems have been devised to synchronise the clocks by making use of these store fill signals. These can be the simple 'single-ended' system, which uses only the signals generated at the local exchange to control the frequency of the clock at that exchange, or the more complicated 'double-ended' system which uses the signals generated at both ends of the line. This latter method introduces the complication of data links, but has many advantages and is more likely to be used in practice. Both classes of system will be discussed in subsequent chapters.

1.2 A Short History of Work on P.C.M. Synchronisation

During the last decade, a large number of papers have been written concerning p.c.m., many of which have been on the subject of 'synchronisation'. This subject, however, is often taken to mean the decoding of the special 'frame start' signals, which is properly known as 'line synchronisation'. We are concerned with the aspect of 'exchange clock synchronisation' as discussed above.

The early work on this subject was undertaken by Bell Telephone Laboratories, Inc., following the initial work by Vaughan in 1959[2] on exchange design. Much other work on exchange design has been done, the papers by Mornet, Chatelon and Lecorre in 1963[4] and Walker and Duerdoth in 1964[3] being among the first.

The work at Bell Laboratories on synchronisation was undertaken by Beneš, who is said to have concluded that it was not possible to synchronise geographically separated oscillators. In 1963, Inose, who had worked under Beneš and still had connections with Bell, started work on this subject at Tokyo University. The initial results were published in 1964[5], in Japanese, and are said to have stimulated further work at Bell Laboratories. No reports of this activity were published until 1966.

The British Post Office was also active in this field, studies being undertaken by Duerdoth and by Jarvis. Duerdoth described a non-linear system at Paris in 1966[6] and Jarvis demonstrated a model of his linear 'double-ended' system[7] at a Post Office open day in 1966. Both of these systems will be described later.

Earlier in 1966, Inose had published the first of a series of papers in English[8] in which the theoretical aspects of a 'single-ended' control system were discussed. This discussed the requirements for synchronisation to be possible, namely that all stations be interconnected in some way, and gave a condition satisfied by the connection matrix when this is so. It gave the final frequency as the ratio of determinants obtained by means of the Laplace final value theorem and Cramer's rule, and the paper also gave an introduction to the problem of showing stability, without actually quoting any criteria for gain versus delay or filter transfer function.. This paper was based on a more complete paper in Japanese for which no translation was available in this country for three years, that is until 1969.

Mumford and Smith in September 1966[9] published a method for making digital corrections to the clock frequency. This was in effect a non-linear single-ended system which has the drawback that the final frequency can assume one of several distinct values and can even be different in different parts of the network. This rather remarkable disadvantage can be overcome in a number of ways, as is discussed in Section 2.3 on non-linear systems.

In October 1966 the work described in the body of this thesis was commenced at the University of Warwick, under the supervision of Mr. P. C. Parks. He suggested an approach using the Gershgorin circle theorem[10] for location of matrix eigenvalues, as the control problem first examined here is a multivariable linear system and the eigenvalues show whether the system is stable or not. A preliminary paper[11] published early in December 1966 showed how this method could be applied. An attempt was made to consider line delays, although more elegant proofs of stability were to follow later, after the submission of the M.Sc. degree dissertation[18].

A group of three papers was published in the Bell System Technical Journal in December 1966. The first of these three, by Gersho and Karafin[12], describes a single ended control system. The system equations are deduced for this model, and various theorems are given for the properties of the connection matrix. A stability condition is given for this type of system which is the basis of the subsequent results for single-ended systems.

The second paper, by Karnaugh[13], discusses the formulation of the system equations in some detail. The equations for the buffer store 'fill' signals are found, and a vector space of phase variables discussed. It is shown that for a particular class of non-linear systems with zero time delays, stability is assured. Some unpublished work of V. E. Beneš is discussed, relating to the stability conditions and final frequency of a linear control system. There are restrictions on the gains and filters used in the Beneš model and the proofs are not given. Karnaugh points out limitations of the Beneš model concerning the formula for final frequency, and deduces an alternative. He concludes that there was no general proof of control system stability, but proofs exist for the stability of his non-linear model without time delays and for the Beneš model. This latter model appears to be a 'single-ended' system with common filters on all paths and the same gain used at every exchange.

The third paper published at this time, by Brilliant[14], discussed the final frequency of the oscillators. After discussion of the initial setting-up of the network, he introduces a number of theorems. One of these is included in the Gershgorin Circle theorem, and applies to 'diagonally dominated' matrices, and another is an incomplete form of that presented in Chapter 4 of this Thesis, which discusses final frequency. Consequently, he gives the final frequency in terms of matrix cofactors, without being able to evaluate it explicitly in terms of the system variables, as is done in this Thesis. He concludes that the final frequency does not depend on the manner in which the network is set up.

In January 1967, Inose, Saito and Fujisaki published a further paper[15] in which an important aspect of the systems, known as 'wrong mode operation', was mentioned. These wrong modes are due to the cyclicity of the phase comparators. It also discussed a possible method for the correction of these modes, although precise details of its operation are not given.

The subject of dynamic system response was raised by Brilliant[16] in February 1967. He discussed the response of a number of special cases, and tentatively suggested that more general results could be inferred. He points out that a fully interconnected network has advantages over others, in terms of transient response, but doubts if this type of network is a practical solution to the problem. He makes no recommendations for a preferred type of network.

In May 1967, a further publication of joint work by Parks and Miller[17] considered the estimation of transients by a Liapunov method. This showed that in a system without delays the transient that could appear at any exchange was bounded by a decaying function of the initial disturbance. The rate of decay was shown to depend considerably on the degree of interconnection of the network, tightly connected networks having the fastest rate of decay.

The results obtained by Miller were summarised in a dissertation for the degree of Master of Science at the University of Warwick in May 1967[18]. In addition to the stability and transient studies already published, expressions for the system final frequency were derived for both the double-and single-ended systems. The system equations were subject to some restrictions which are removed in this Thesis.

In June 1967, Inose, Fujisaki and Saito[19] expressed the problem of wrong mode operation in mathematical form. For a network of n exchanges they considered sets of determinants of order n. An example is given for the case of three exchanges fully interconnected, but no general conclusions are reached.

West, in a paper published in October 1967[20], discussed a single-ended linear control system. He deduced a set of equations and found the final frequency of his system. He considered a special case and the formula thus obtained is a less general form of that given in this Thesis. In the form presented in the paper, it is not made clear that the operation mode can affect the system frequency, although a term corresponding to this appears in the formula. Stability is discussed for a special case with zero time delays by means of the Gershgorin theorem. This is exactly the same method that had been published earlier by Parks and Miller[11]. A more general case with time delays was also presented, which appears to be the proof of the Beneš condition given in the paper by Karnaugh[13]. He concludes that the system frequency will vary with the line delays more than the controllable range of the oscillators will allow, and that the insertion of limits on the oscillators will cause some exchange to act as a master reference.

A third joint paper by Parks and Miller[21], published in January 1968, again considered system stability. This paper showed that the 'diagonal dominance' method used by West was in fact a special case of the Gershgorin theorem. The treatment of the single-ended system was extended to the case of general values of gain, completing the proof of the Beneš result. A stability condition for a double-ended system, better than that given earlier[18], was given without proof. The derivation of this result appears in a later chapter of this Thesis. Exact stability conditions for systems with filters were not given.

Hartmann, working at Siemens, Munich, published a short account[22] of some experimental work in January 1968. He obtained a condition for the stability of a single-ended system with delays and filters, which is in agreement with the theoretical predictions of this Thesis, as published[36] in July 1969. This condition is that for a system with low-pass filters in the feedback paths, the product of total gain and filter time constant at any exchange shall be less than ½. The practical work was done with a network of five exchanges.

Saito, in February 1968[23], continued the discussion of wrong mode operation. He examined the variation of system frequency with operating mode and obtained general conditions for the existence of wrong modes, in terms of an equation which must be satisfied by the state vector. He did not show how the wrong mode can be detected, but stated that it would be possible to keep a network in the desired mode in which the buffer storage is minimised.

Yamato, Ono and Usuda published two papers[24, 25] describing a double-ended system. The first, published in February 1968, is a shortened version of the later paper of June 1968. The papers describe a practical experiment, which uses the double-ended technique, and also contain some theoretical work. They obtain an expression for system frequency, which is the same as that previously given in the M.Sc. dissertation[18], showing that the frequency is independent of line delays or network configuration. They do not consider the effects of filter time constants, which are included in this analysis by Miller. They derive a stability criterion for a network of n exchanges fully interconnected with double-ended control. This is correct for up to four exchanges, the size of their experiment, but does not apply to larger networks. This limitation is not mentioned in their paper, but is discussed by Parks and Miller in their I.F.A.C. Symposium paper[29], which gives the complete condition.

A double-ended system is also described by Candy and Karnaugh in their paper of February 1968[26]. They consider systems with different gains in the local and remote buffer store signal channels. When the gains are equal, this is the double-ended system proposed by Jarvis, known as a balanced system, which is very likely to be used in practice. When the gain applied to the remote path is zero, the system reduces to the single-ended system. In the paper, they consider the sensitivity of the system frequency to parameter changes and also the system stability. They quote a condition, without proof, attributed to Brilliant. Examining the situation of unbalanced systems in the manner used to derive the result of Parks and Miller[21], it can be seen that the result of Candy and Karnaugh is a simplification of the general sufficient condition, and as such implies a limitation of the gain-delay product. The result of Parks and Miller had already given the better condition for the particular case of a balanced control. The general analysis of this situation is given in Section 3.2.2 of this Thesis. It will also be shown in Section 4.2 that the unbalanced system is somewhat inferior and unlikely to be used in practice on grounds of variation of system frequency with delay. Candy and Karnaugh also consider the use of certain types of non-linearity, but conclude from their experiments that there is little point in including such devices.

The problem of phase modes was considered by Parks and Miller in a paper published in June 1968[27]. The general matrix equations were discussed, and it was shown that in a network of n exchanges it is possible to express the state of the network in terms of a vector $n - 1$ principal phase differences, chosen such that the phase of each of the n exchanges is included. In the special case of full interconnection, the matrix connecting the steady state phase differences to the non-linearities in the phase comparators reduces to the unit matrix, allowing simple results to be obtained. In this paper the case of a three exchange system is discussed, and the initial frequency differences and delay variations are shown to affect the final phase differences.

The I.F.A.C. Symposium paper[29] of October 1968 by the same authors contained, in addition to a summary of their previous papers, some additional work on these wrong modes. This considered in particular the case of four exchanges fully interconnected. In this case no wrong modes are possible, and in fact all networks of four exchanges have this property. The incomplete result of Yamato et al.[25]

for the stability of fully interconnected systems was mentioned, and the complete result given. Unfortunately there was an error in the derivation, but the sufficient condition is still correct.
The formulae for the system frequency of single- and double-ended systems were given, with some derivation.

Hartmann, in September 1968, had published a further paper in German[28]. This expanded his earlier paper[22] and discussed a series of experiments and associated mathematics. He discussed the factors influencing final frequency, and obtains a formula for the frequency which depends upon the matrix cofactors, as in the paper of Brilliant[14]. He derives the stability condition quoted in the earlier paper and also shows some experimental results obtained from tests of the transient response of different networks. These confirm the stability result.

Hills, in a paper published in December 1968[30], discusses a single-ended system. He uses a modified form of proof, similar to the earlier proofs, to show system stability. The approach is based on filter theory, designed to give insight to the less specialised engineer. The stability condition resulting is the same as that of Hartmann[22] and others. The effect of parameter changes is discussed, and as already mentioned, it is found that the single-ended control system is sensitive to most parameters. He also shows that the initial conditions affect the system frequency. This is because the operation mode depends upon the initial phase differences, although this is not stated explicitly. Thus for a two station network, as in Hills' example, there are two distinct frequencies as there can be two possible phase modes. This paper, while instructive to the non-specialist, does not contain much new material, presumably because of the long time delay between completion and publication of the work.

The most recent paper from the group working at Bell Laboratories was published by Pierce in March 1969[31]. He considers the problem of synchronising high-speed pulse streams such as those generated by high order multiplexes. The paper is largely descriptive of system philosophy, with the development of an analogous set of linear network equations. He considers a generalised system with provision for the frequency deviation or its first derivative to be controlled. In the latter case, the oscillator frequencies as well as the store fills will be error variables. He argues that time delays can be disregarded as all corrections will be slow compared with the transmission time of the lines. He gives an expression for the final frequency in terms of the initial fills of the buffer stores which are used as phase detectors. This expression does not indicate that the store fills are themselves functions of oscillator phases and store overflows. When this is done[18], it can be seen that only the store overflows affect the final frequency. The effect of line delay is not mentioned in this context. He mentions that some approaches to the system design use variable delay devices at the incoming terminals to minimise buffer storage. This is not proceeded with, but it is well known that a system of this type has the effect of reducing the effective gain of the control system if the output of the variable delay is used as an indication of distant phase. By means of his analogue circuit, Pierce shows that a single-ended system without delays is always stable, even in the presence of non-linear gain functions. This is a generalisation of the argument of Hills[30]. In spite of this, Pierce indicates limits for the system gain. An upper bound is given by the desire to limit the amount of correction given to what are basically stable oscillators and a lower bound by the condition that enough gain must be available to drive the oscillators to a common frequency without exceeding the buffer storage. He suggests that separate, presumably manual, adjustments should be made to compensate for the slow changes in buffer storage due to delay changes.

Pierce does not discuss the stability of systems with delays, but refers to a paper to be published shortly by I. W. Sandberg, of Bell Laboratories.

Work on the dynamic response of systems is discussed in a paper by Saito and Yasuda[32], published in April 1969. They consider only a fully interconnected single-ended system with delays and filters and give an analytic solution for this case. They conclude that the effects of the line delays and repeaters are small. An illustration is given of the effect of a frequency step in a three exchange system of this type. Also discussed is the problem of pull-in range of the oscillators. This is associated with the non-linear nature of the phase comparators. If a comparator passes the end of its linear range, a wrong mode will result and the phase differences will not then be sufficient to hold the oscillators at a common frequency. This will occur if the range of uncontrolled frequencies is too large for the gains used, so that the transient of station acquisition exceeds the store limits. For the case of critical damping, a figure illustrates this.

A further paper by Saito[33], read at the IEE conference on telephone switching systems in April 1969, summarises his work on wrong mode operation[19, 23]. He here describes in detail the procedure he advocates for the suppression of the wrong modes. This is to make certain exchanges lock to the phase of certain other exchanges in a manner dependent on the network configuration and particular mode. This has the effect of changing the network configuration such that when the network is restored the desired 'in-phase' mode is selected. No indication is given of any general solution to this problem; this method appears to depend on detailed knowledge of the behaviour of the given network and all the possible sub-networks resulting from the same number of exchanges being less connected.

In all the above papers it has been assumed that the operation of the control systems can be approximated to a continuous, usually linear, set of equations. Rees, in another paper[34] presented at the above conference, described a system which is definitely discontinuous. The corrections are based on the values of store fills at certain sampling intervals, which are chosen to be separated by more than the longest system time constant. The gains used are subject to somewhat arbitrary limits based on the concept of slow changes. A possible advantage of this system is that data links can be used to supplement the basic network, so that any network can appear to be fully connected for control purposes. A possible disadvantage is the difficulty of combining the error signals sufficiently accurately. This feature is available as the sampling period is longer than the maximum loop delay between any two exchanges, via intermediate exchanges if necessary.

An analytical approach to the stability of these sampled systems was published by Miller in May 1969[35]. This used the Z-Transform of the system equations and applied the usual stability criterion that there shall be no roots of the characteristic equation outside the unit circle of the z-plane. This paper derives results for single-ended systems with delays and filters and double-ended systems with delays. It shows that as the sampling period is increased the gains must decrease to maintain stability. The case of double-ended systems with filters as well as delays is not covered in the paper, although it will be discussed in chapter 3 of this Thesis.

A further paper by Miller[36], published in July 1969, describes both single-and double-ended systems. This applies the continuous system analysis to determine the factors affecting stability and final frequency of the systems. The stability results for the single-ended system follow the methods of West[20] but express the result in terms of gains which are different for each input to an exchange. Filters are included in the analysis and shown to limit the gain constants. Also included is the analysis of the double-ended system with filters. This extends the result of the earlier paper[21]. The final frequency analysis applies to double-ended systems which may be unbalanced. An extreme case of this is the single-ended system, so the result applies to both systems. This result shows that a balanced double-ended system is preferable as the frequency is a function of initial frequencies and gain constants only. The other systems all have terms affected by line delay and operation mode. The problem of operation modes is also discussed, although no mathematical detail is given. A method for releasing a network from an unwanted mode is suggested; the network is arranged so that on receipt of a special signal each exchange synchronises to a nominated parent only. Details are given of two methods for the detection of these modes to initiate the special action until the correct mode is found. All of the work published in this paper is discussed in greater detail in this Thesis.

The I.F.A.C. paper of Parks and Miller[29] has been revised to remove errors and is published in the I.F.A.C. journal "Automatica" in July 1969[37]. This paper gives a mathematical treatment of the mode problem and a summary of the work completed up to May 1968. It also contains some additional information on the problem of setting the line delays, and hence exchange phases, when creating the network.

2 CONTROL SYSTEMS FOR P.C.M. SYNCHRONISATION

There have been a large number of suggested systems for controlling the network frequency, and these can be divided into a number of categories. The first distinction is between 'single-ended' and 'double-ended' controls. In the former, the exchange oscillator frequency is a function of the buffer store fills at the incoming line terminals at that exchange. In the latter, each line system is arranged to signal the buffer fills to the remote ends of the lines so that the oscillator frequency is a function of both remote and local fills of the line systems directly connected to the exchange. In most cases, the control is arranged to be continuous and linear. Some non-linearities have been suggested, but it is difficult to treat these theoretically. Indeed, the only work completed so far in this direction, by Candy and Karnaugh[26], suggests that there is little advantage in such non-linearities.

In actual fact all the systems are discontinuous to some extent. This is due to the action of the phase detector, which acts as a sampler. The sampling frequency is usually high, but certain proposals, such as the system devised by Rees[34], uses a low sampling frequency quite deliberately. Thus it is important to discuss this type of system.

One class of non-linear system will be discussed. This is that favoured by Duerdoth, where no action takes place until a buffer is near its full or empty state. The analysis of this class is again difficult, but some discussion is possible.

2.1 Continuous Linear Systems

In these systems the buffer fill is used as a measure of phase difference between the two stations. The buffer store is, typically, a number of bistable devices, set in turn by the incoming line digits and read-out into the exchange under the control of the local clock. To organise the writing process, an incoming retimer clock is provided. This is exactly the device used to demodulate the signals on a point-to-point link. An oscillator is forced to run at the mean frequency of the line message; this device is used at every digital repeater also. The output is used to drive a number of counters to indicate the slot and bit currently being received and certain line pulses are checked to ensure that they are always in a particular pattern. This pulse pattern is inserted at the sending end in some signalling bit positions. If it is not detected the incoming clock assumes line synchronisation to be lost and alters the counter state until it is. This process of 'line synchronisation' is the subject of many specialised papers.

The phase comparator is usually another bistable device set by a particular digit pulse from the incoming clock. This could be the first pulse of the first slot in a frame. If the bistable device is reset by the corresponding pulse of the local exchange clock, the output waveform is a function of the 'fill' of the store. However, we wish to bias this signal so that with the two clocks in synchronism a unity mark-space ratio is produced. Thus the device is set by the first pulse of the frame and reset by the $(\frac{n}{2} + 1)$th pulse which would be the first pulse in the 13th slot of a 24 channel system. Waveforms are shown in fig. 2.1.

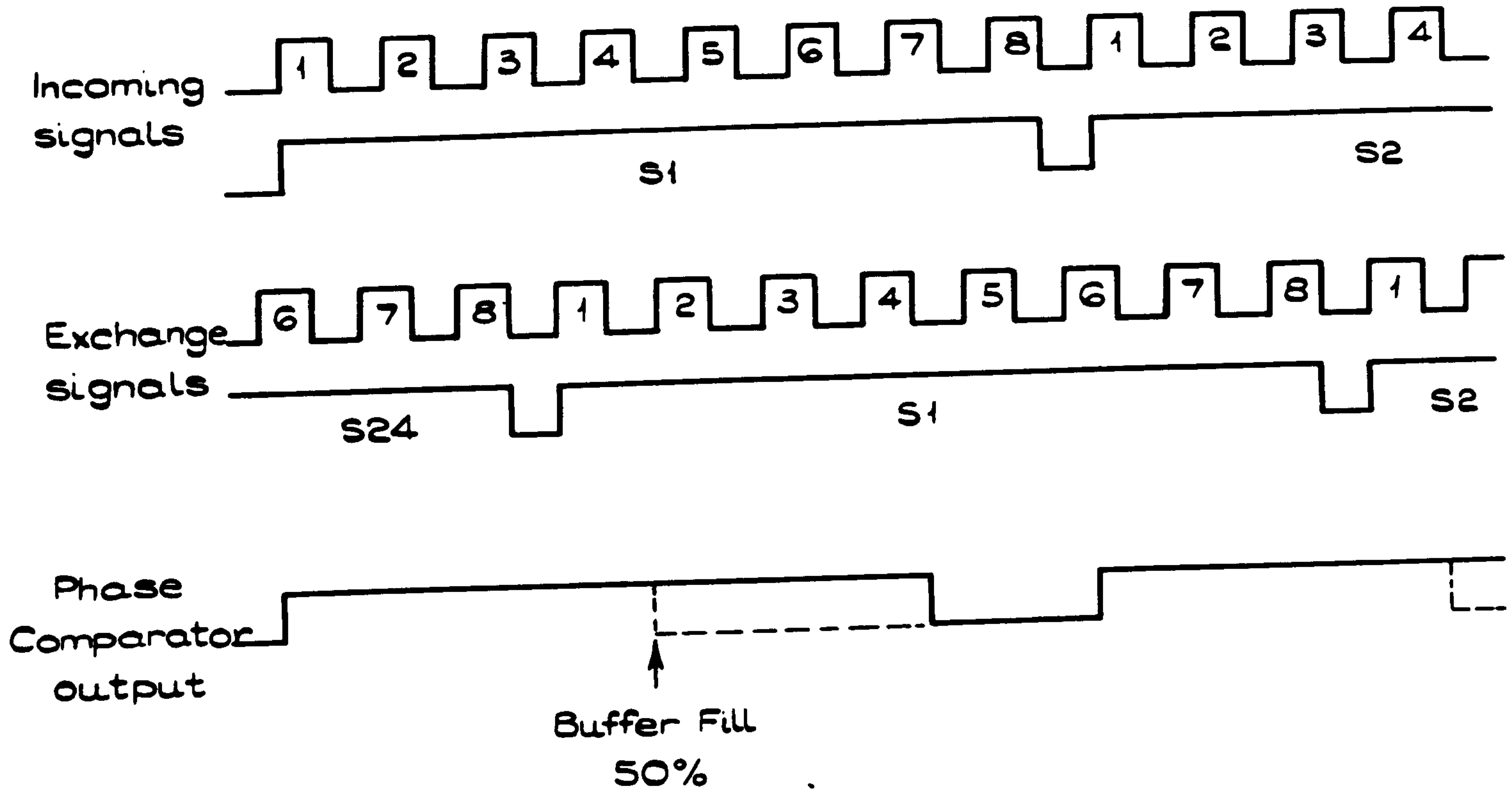
Such a phase comparator has three main drawbacks. The first is that the signal is available only once per frame and the control system is therefore a sampled system with frequency 8 kHz. By means of 'fine' and 'coarse' comparators it is possible to increase the effective sampling frequency to the slot rate, 192 kHz. The second drawback is that the comparator is cyclic and cannot detect differences of more than half a frame either side of the reference local phase. Any output could thus be in error by an integral number of frames. This causes the phenomenon of wrong mode operation, due to false interpretation of phase differences by the control system. The third problem is that the output is the apparent phase difference which depends upon both the true phase difference and the line delays. This drawback is largely eliminated by the double-ended system.

The symbol $\phi_{ij}(t)$ denotes the apparent phase difference measured at time t at the ith exchange, which operates at frequency $f_i(t)$, by comparison with the signals from the jth exchange, operating at frequency $f_j(t)$. These signals have passed through a line with a total delay of d_{ij} seconds and through a number of regenerative repeaters. The transfer function $G_{ij}(s)$ is used to denote the cumulative effect of these repeaters and the terminal clock recovery circuit.

The apparent phase difference is given by

$$\phi_{ij}(t) = \{\phi_{oj}\}_{Gij} + \int_0^{t-d_{ij}} \{f_j(\gamma)\}_{Gij} d\gamma - \phi_{oi} - \int_0^t f_i(\gamma) d\gamma \qquad \ldots \quad (2.1.1)$$

$\{f_j(t)\}_{Gij} = f_j(t) * g_{ij}(t)$, where the * denotes convolution,



Phase comparator set by pulse P_{1L}, reset by pulse P_{5E}. Period of output characteristic is one slot. If slot pulses S_{1L} and S_{13E} are used, period becomes one frame. (see also Fig. 5.1)

FIG. 2.1. OPERATION OF PHASE COMPARATOR

denotes the signal $f_j(t)$ after it has passed through the filter with transfer function $G_{ij}(s)$. The phase comparator output is given by

$$e_{ij}(t) = \phi_{ij}(t) - r_{ij}(t) \qquad \ldots \quad (2.1.2)$$

where $r_{ij}(t)$ is the integer such that

$$r_{ij}(t) \leq \tfrac{1}{2} + \phi_{ij}(t) < r_{ij}(t) + 1 \qquad \ldots \quad (2.1.3)$$

(Note that this definition of $r_{ij}(t)$ is different from that in references 36 and 37.) The output, therefore, is linear between $-\frac{1}{2}$ and $+\frac{1}{2}$, at which limits the counters recycle (see fig. 5.1).

The phase comparator output signal is passed through a low-pass filter to the local oscillator control circuit and, in the double-ended system, over a data-link to the remote end of the p.c.m. line system.

2.1.1 The single-ended system

Each phase comparator output signal is passed through an individual filter and amplifier before being added to the other signals to form the frequency correction signal as shown in fig. 2.2. The oscillator frequency at the ith exchange is thus

$$f_i(t) = f_{oi} + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\{e_{ij}(t)\}_{F_{ij}} \qquad t \geq o \qquad \ldots \quad (2.1.4)$$

where f_{oi} is the natural frequency of oscillation without the control and the gain k_{ij} is zero unless a direct path exists between the ith and jth stations. For $t < o$, $f_i(t) = f_{oi}$. The filter with transfer function $F_{ij}(s)$ is that used to smooth the phase comparator output.

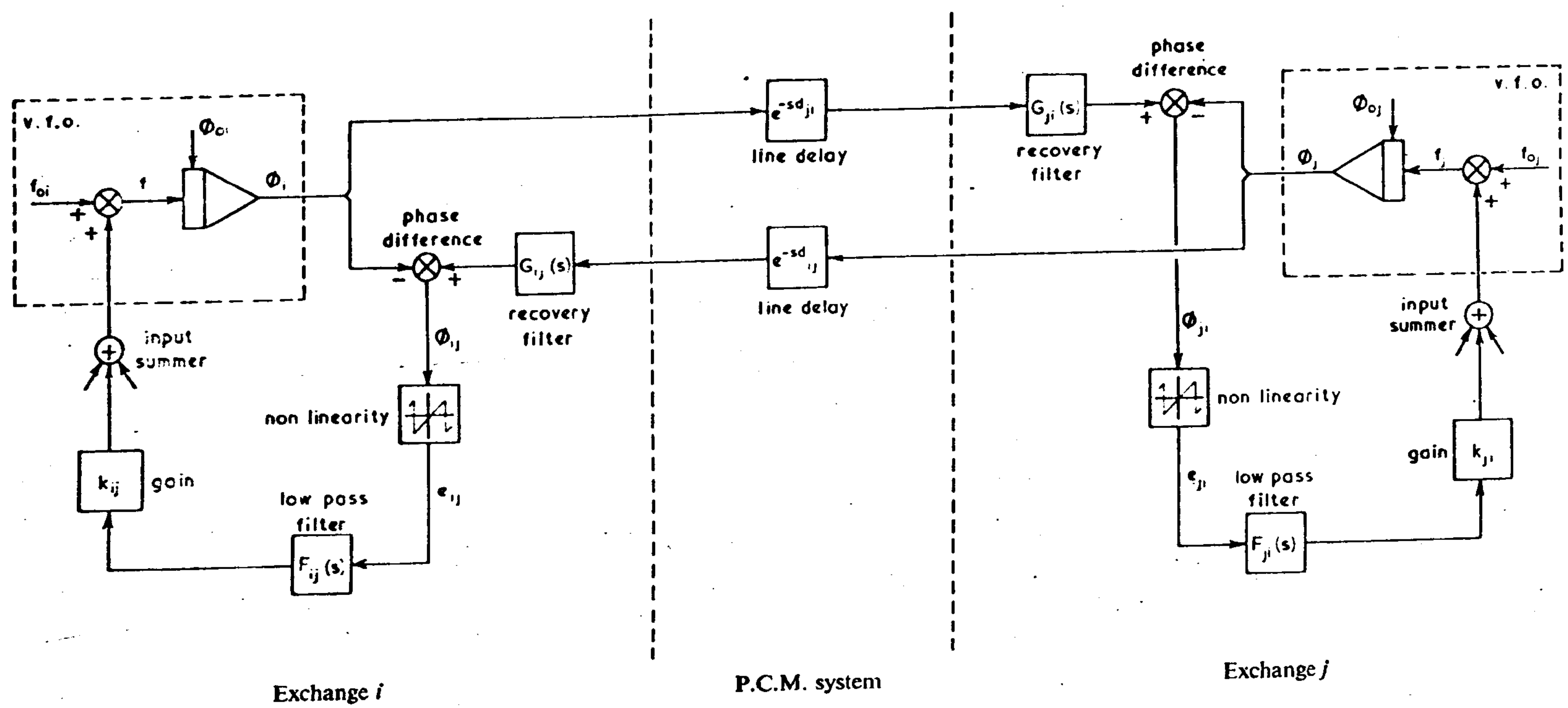


Fig. 2·2. Single-ended control system.

In some systems the gains or filters used at an exchange will be common to more than one input. However, wherever possible we will consider non-identical values on all inputs.

Single-sided Laplace Transforms of the equations *(2.1.4)* are formed, after substituting for $e_{ij}(t)$. $\overline{f}_i$ denotes the transform of $f_i(t)$; this is, of course, a function of s. Thus, for $i = 1, 2, \ldots n$

$$\overline{f}_i \{s + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}F_{ij}(s)\}$$

$$= f_{oi} + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}F_{ij}(s)G_{ij}(s)e^{-sd_{ij}}\,\overline{f}_j + u_i(s) \qquad \ldots \quad (2.1.5)$$

where $u_i(s)$ is defined by

$$u_i(s) = \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}F_{ij}(s)$$

$$\left\{+ \phi_{oj} - \phi_{oi} - r_{ij} - G_{ij}(s)\left(\frac{f_{oj}}{s}\right)\left(1 - sd_{ij} - e^{-sd_{ij}}\right)\right\} \qquad \ldots \quad (2.1.6)$$

The transforms are only valid if the variables $r_{ij}(t)$ are treated as being constants, r_{ij}. They describe the behaviour of the system in a region for which all the phase comparators are in a linear state and do not overflow. Should overflow occur, the system acts in a piece-wise linear manner and the transform is not valid. The end states are, however, the same, apart from the change in one or more of the r_{ij}, so that it is possible to trace the motion from one linear region to another by inverting the transforms. This is discussed further in the chapter on wrong mode operation.

The equations (2.1.5) for the frequencies of the n exchanges may be written in matrix form

$$[s\underline{I} - \underline{A}(s)]\,\underline{F} = \underline{f}_o + \underline{u}(s) \qquad \ldots \quad (2.1.7)$$

where $\underline{I}$ is the unit matrix and $\underline{A}(s)$ is defined by its elements

$$a_{ii}(s) = -\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}F_{ij}(s)$$

$$a_{ij}(s) = +\,k_{ij}F_{ij}(s)G_{ij}(s)e^{-sd_{ij}} \qquad (i \neq j)$$

The stability of the system depends upon the matrix $\underline{A}(s)$ and the final frequency upon the vectors $\underline{f}_o$ and $\underline{u}(s)$ in addition. Later chapters are devoted to the discussion of these important aspects of the system design.

2.1.2 The double-ended system

As stated above, it was realised at an early stage that the system frequency of a single-ended system would depend in some manner, then unknown, upon the line delays which are temperature dependent. It was felt that this was unacceptable, especially with possible international digital connections, so the double-ended system was devised by Jarvis[7]. In his prototype system he used a type of delta modulation to pass the buffer store fill signals over the p.c.m. system. The modulator sends bits over the line to be recovered by an integrating device. A similar integrator produces a local copy of the output which is compared with the input signal. Bits of value 1 are sent when the input exceeds the output, which otherwise decays. The time constant has to be chosen so that the rate of decay exceeds the fastest negative rate of change of the input signal. A single bit every frame was found to give a useful bandwidth of 1 Hz after filtering. Other data link systems have been suggested that use a lower line bandwidth than the 8 kBit/s needed here at the cost of some extra delay. It will be seen that it is this data-link delay that limits the value of the gain constants that may be used in a stable system. Thus the design of a good data-link modulator is central to the proper operation of this system. The filter time constant also affects stability, so to maximise allowable gain the delay and filter time constants must both be optimised, as discussed in the next chapter.

The data-link for transmitting the buffer store fill of the link from i to j back to exchange i has a pure delay of τ_{ij} and an associated filter with transfer function $H_{ij}(s)$. As shown in fig. 2.3, the remote signal is subtracted from the local signal before amplification and addition. The oscillator frequency at the ith exchange is now

$$f_i(t) = f_{0i} + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\left[\{e_{ij}(t)\}_{Fij} - \{e_{ji}(t - \tau_{ij})\}_{Hij}\right]$$

$$\text{for } i = 1, 2, \dots n, \quad t \geq 0. \qquad \dots \quad (2.1.8)$$

Again it is assumed that the phase comparators remain in a linear region and Laplace Transforms are formed to obtain the matrix equation

$$\left[s\underline{I} - \underline{B}(s)\right]\underline{\bar{f}} = \underline{f}_0 + \underline{v}(s) \qquad \dots \quad (2.1.9)$$

where

$$b_{ii}(s) = - \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\{F_{ij}(s) + G_{ji}(s)H_{ij}(s)e^{-sd_{ji}-s\tau_{ij}}\}$$

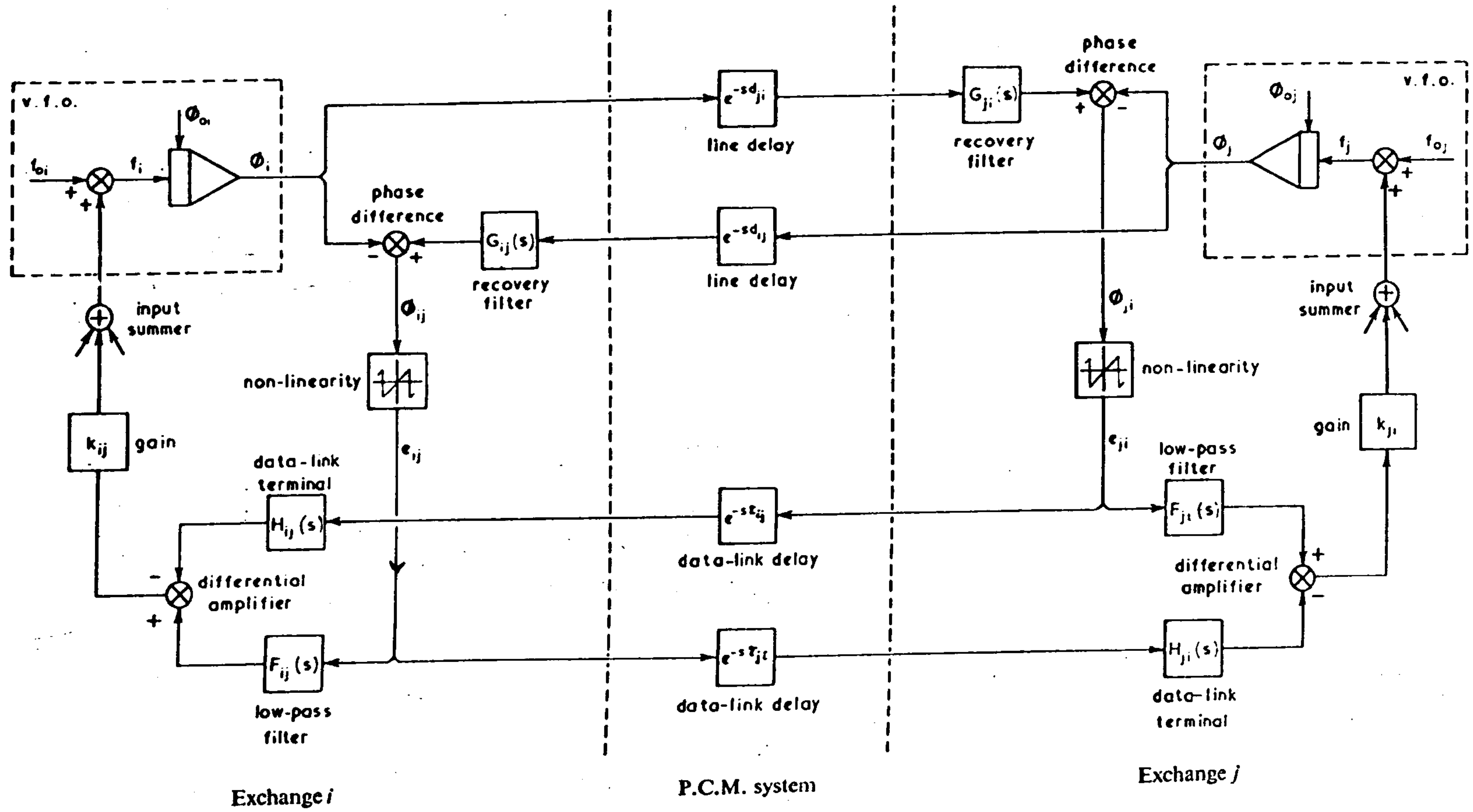


FIG. 2·3 Double-ended control system.

$$b_{ij}(s) = + k_{ij}\{F_{ij}(s)G_{ij}(s)e^{-sd_{ij}} + H_{ij}(s)e^{-s\tau_{ij}}\} \qquad (i \neq j)$$

and

$$v_i(s) = \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\Bigg[\{F_{ij}(s)G_{ij}(s) + H_{ij}(s)\}$$

$$\{\phi_{oj} + \frac{f_{oj}}{s}\left(1 - sd_{ij} - e^{-sd_{ij}}\right)\}$$

$$- \{F_{ij}(s) + G_{ji}(s)H_{ij}(s)\}$$

$$\{\phi_{oi} + \frac{f_{oi}}{s}\left(1 - sd_{ji} - s\tau_{ij} - e^{-sd_{ji}-s\tau_{ij}}\right)\}$$

$$- \{r_{ij}F_{ij}(s) - r_{ji}H_{ji}(s)\}\Bigg]$$

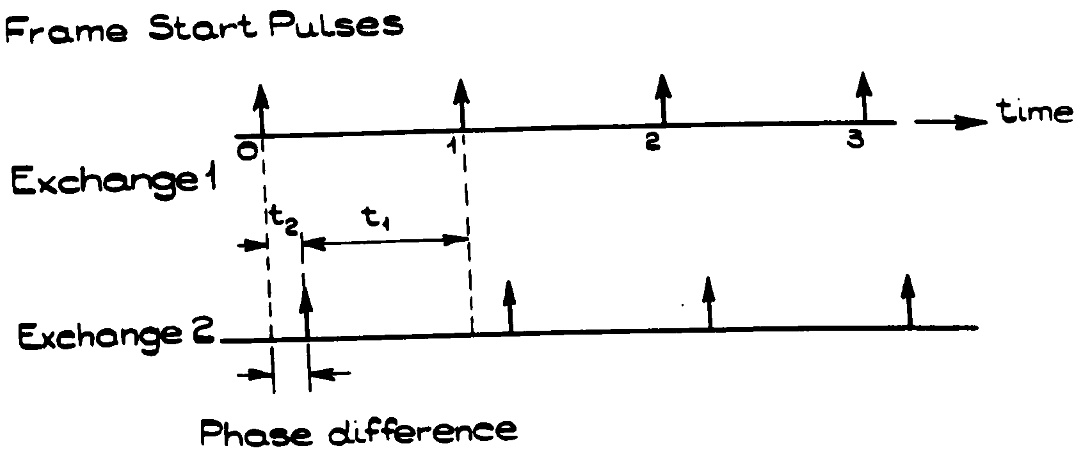
As for the single-ended system, the matrix $\underline{B}(s)$ determines the stability and the vectors $\underline{f}_o$ and $\underline{v}(s)$ determine the final frequency. These equations do not, however, reduce to the single-ended system equations by putting $H_{ij}(s) = 0$ and $\tau_{ij} = 0$ for all i, j; the term in f_{oi} of $v_i(s)$ does not appear in $u_i(s)$ as it refers to the initial conditions of the fed-back buffer store signal. However, when considering small values of s, as in the derivation of the final frequency expression, all the terms in $\underline{f}_o$ disappear from both $\underline{u}(s)$ and $\underline{v}(s)$. For this particular case it is possible to deduce the result for the single-ended system from the double-ended system result.

2.2 Sampled Systems

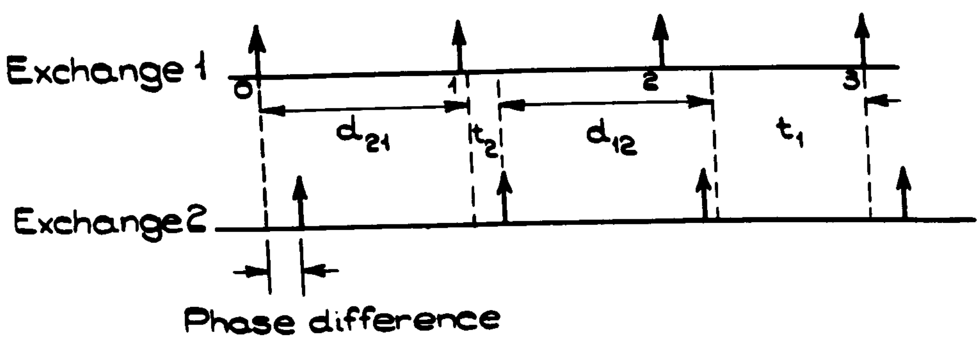
All the systems described in section 2.1 are in fact sampled although it may generally be assumed that the sampling frequency is sufficiently high for a continuous system analysis to be valid. However, certain difficulties with the data-link of the double-ended system have suggested that a low sampling speed will be used. This may only apply to a part of the system, in which case a multi-rate system will result, but we will here assume that all sampling is at a common frequency. This cannot be higher than the working frequency of the phase detector, which we will assume to function at frame rate (8 kHz) or less. It is likely that the data-link for the double-ended system will send a group of pulses every 10 frames; this group will be transmitted a pulse at a time but no action can be taken until the whole group is received. Certain registers will be provided to hold samples of buffer store fill for the duration of the sampling period. These registers will ensure that sampling may be regarded as simultaneous at every exchange although it will actually occur at a certain phase of each individual clock. At this phase all the samples received from other exchanges will be combined and a frequency correction made. The correction will be held until the next sampling instant. The new value of this frequency will become apparent at distant exchanges after a delay due to the line transmission time and the interval between samples at that distant exchange. This will cause a further frequency correction which will be held for a number of sampling periods and become apparent at the first exchange at some later sampling instant. When the sampling phases are not identical at two exchanges, there will be a delay of at least one sampling period between a correction being made and its effect at other exchanges being felt. This delay will be increased by the action of line delays, as indicated in fig. 2.4.

Filters will be needed in a practical system to reduce the effects of noise. It must be remembered in the system design that any random phase variations, or 'jitter', produced by the line repeaters will be present in the buffer store fill signals. The buffers will hold the information until it is read at the receiving exchange clock time. Thus the storage time will fluctuate randomly in a complementary manner to the apparent line phase. This jitter will have mainly high frequency components, but the sampling inherent in the phase comparator will cause all this jitter to appear in a frequency spectrum with bandwidth a half of the sampling frequency. We should thus choose a high sampling rate for the phase comparator and apply a filter to eliminate all frequencies above the pass band of any subsequent low-speed samplers. It is not possible to filter out the jitter before the phase comparator. In the model used to examine the effects of sampling on system stability, we assume the use of certain filters. These are shown in the block diagram of fig. 2.5. The use of digital filters is also a possibility in a practical system, although those considered in the analysis are of the conventional analogue low-pass type. This is a result of the study being related to a particular proposed system.

The block diagram also shows certain fixed signals, such as uncontrolled frequency. Terms corresponding to these will appear in the expression for final system frequency, which is not derived here because of the limitations of the model. The frequency recovery circuit transfer function $G_{ij}(s)$ should also be considered in a full analysis; it will be neglected here to reduce the complexity of the analysis.



(a) No Delays



(b) With Delays d_{12}, d_{21}

Note

t_1, t_2 = time information held at E_1, E_2 respectively

d_{12}, d_{21} are line delays.

FIG. 2.4. DELAYS IN SAMPLED SYSTEM.

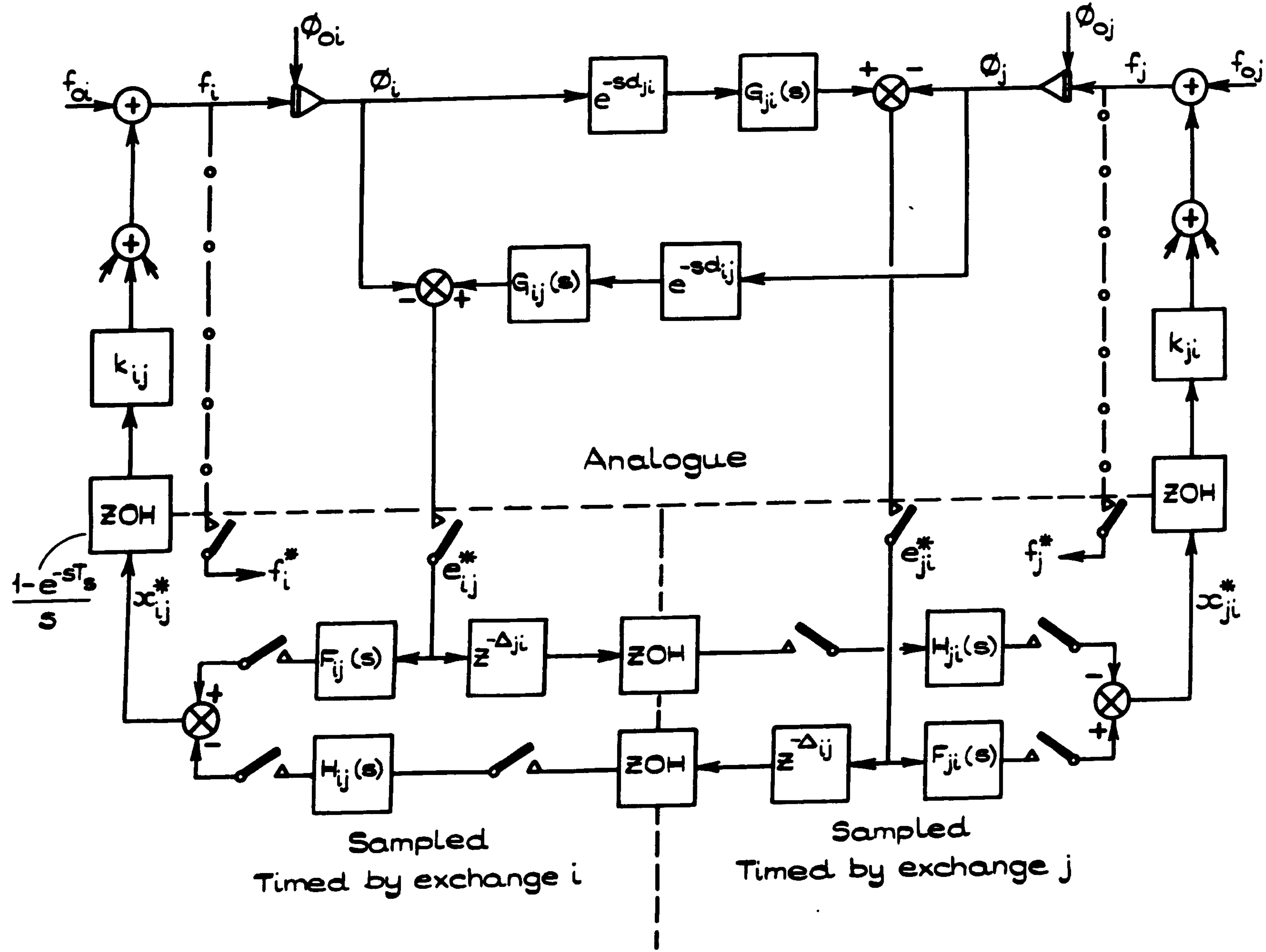


FIG. 2.5 SAMPLED SYSTEM BLOCK DIAGRAM.

The system equations are derived in terms of the Z-transform, where z is defined by $z = e^{+sT_s}$ and T_s is the sampling period. The elements of the block diagram can be rearranged as in fig. *2.6* and the pulse transfer functions deduced.

The oscillator will be treated as a simple voltage controlled device. In practice it is more likely to be a highly stable device with two or more distinct frequencies and obtain intermediate frequencies by high-speed switching from one to another. The block diagram thus shows a zero-order hold followed by an integrator, the latter representing the oscillator. The sampled form of the output phase, $\phi_i(z)$ can now be deduced in terms of the frequency correction signals applied to the oscillator, $x^*_{ij}(z)$. Thus for $i = 1, 2, \ldots n$, $j \neq i$

$$\phi^*_i(z) = \frac{T_s}{z-1} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} x^*_{ij}(z) + \phi_{oi} \frac{z}{z-1} + f_{oi} \frac{z}{(z-1)^2} \qquad \ldots \quad (2.2.1)$$

The signal received at the distant end of the line is modified by the frequency recovery filter, as well as delayed. The delay can be commuted with the filter for analytical purposes. The sampled form of the undelayed modified signal, $\phi_i(s)G_{ji}(s)$, is denoted by $\phi^*_{Gji}(z)$. The filter transfer function, for a first order filter, is $G_{ji}(s) = 1(1 + sT_{Gji})$ so that

$$\phi^*_{Gji}(z) = \frac{T_s}{z-1} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} x^*_{ij}(z) + \frac{z-1}{z} \sum_{\substack{j=1 \\ \neq i}}^{n} \frac{k_{ij} \quad T_{Gji}}{z - e^{-T_s/T_{Gji}}} x^*_{ij}(z)$$

$$+ \phi_{oi} \left\{ \frac{z}{z-1} + \frac{z}{z - e^{-T_s/T_{Gji}}} \right\}$$

$$+ f_{oi} \cdot \left\{ \frac{z}{(z-1)^2} + \frac{z \quad T_{Gji}}{z - e^{-T_s/T_{Gji}}} - \frac{zT_{Gji}}{z-1} \right\} \qquad \ldots \quad (2.2.2)$$

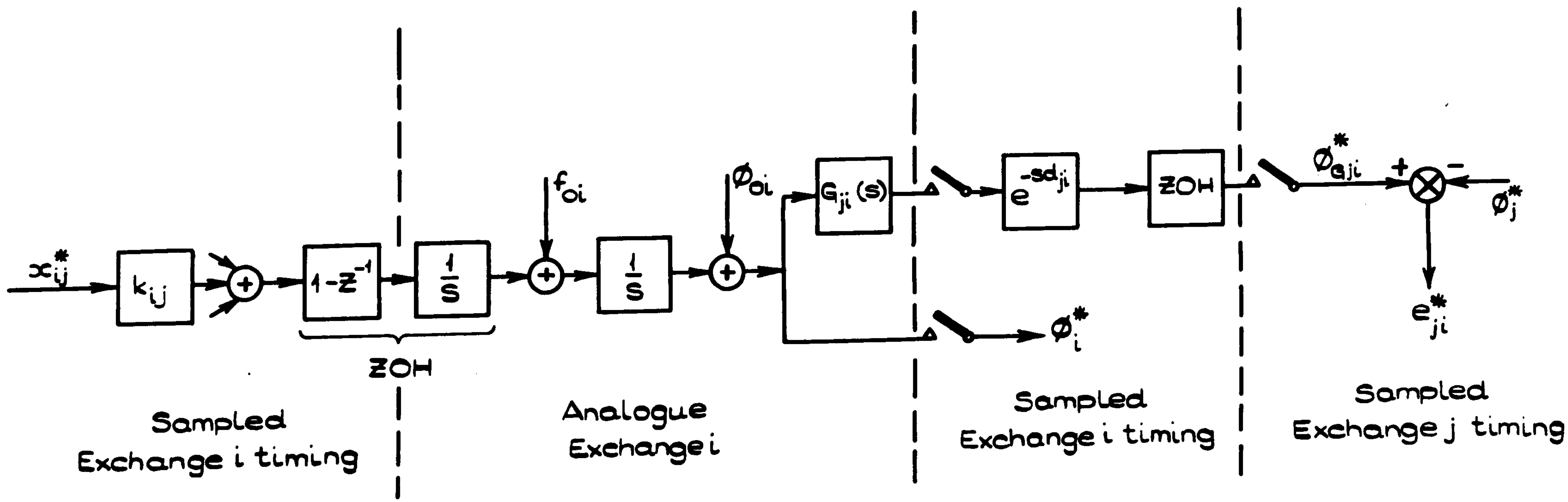


Fig. 2.6. Modified block diagram of sampled system showing flow of correction signal x^*_{ij}.

$T_{Gji} \ll T_s \ll 1$, so that this signal may be approximated to $\phi^*_i(z)$. This is done in equation (2.2.3) and the subsequent analysis.

As shown in fig. 2.4, the line delays can be regarded as being integer multiples of the sampling period. They will have transfer functions such as $z^{-D_{ji}}$, which denotes the line delay between exchanges i and j. The use of this transfer function for the stability analysis neglects the fact that there is a difference between the true sampling instant and the time at which the sample is used. In equation (2.2.2) this would require an amount $f_{oi}(D_{ji}T_s - d_{ji})$ to be subtracted from the ϕ_{oi} to correct for the effect of this when discussing steady state conditions.

Henceforth, the sampled form of the output at the distant exchange will be regarded as $\phi^*_i z^{-D_{ji}}$, giving the sampled form of the buffer store fill signal as

$$e^*_{ij}(z) = \phi^*_j(z).z^{-D_{ij}} - \phi^*_i(z) \qquad \ldots \quad (2.2.3)$$

The frequency correction signal from this p.c.m. link is, therefore, given by

$$x^*_{ij}(z) = F^*_{ij}(z).e^*_{ij}(z) - H^*_{ij}(z).z^{-\Delta_{ij}}.e^*_{ji}(z) \qquad \ldots \quad (2.2.4)$$

The delay term refers to the data-link from exchange j to exchange i.

The delays are related to the continuous system values by

$$D_{ij} = \left[1 + {}^{d_{ij}}/T_s + ({}^{\phi_j}/f_j - {}^{\phi_i}/f_i)/T_s \right] \qquad \ldots \quad (2.2.5)$$

$$\Delta_{ij} = \left[1 + {}^{\tau_{ij}}/T_s + ({}^{\phi_j}/f_j - {}^{\phi_i}/f_i)/T_s \right] \qquad \ldots \quad (2.2.6)$$

using the Gaussian bracket notation that $[n]$ is the integer such that $[n] \leq n < [n+1]$. The phase terms in these expressions arise from the fact that with zero delays it is still possible to have a delay in the receipt of signals relating to a frequency correction at an adjacent exchange, as illustrated in fig. 2.4.

The expression for $f^*_i(z)$ is derived by considering an additional sampler to be measuring $f_i(t)$. By means of equations (2.2.1), (2.2.3) and (2.2.4) it can be seen that for $i = 1, 2, \ldots n$ and $j \neq i$

$$\left\{ 1 + \frac{T_s}{z-1} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\{F^*_{ij}(z) + H^*_{ij}(z).z^{-(\Delta_{ij}+D_{ji})}\} \right\} f^*_i(z)$$

$$- \frac{T_s}{z-1} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\{F^*_{ij}(z).z^{-D_{ij}} + H^*_{ij}(z).z^{-\Delta_{ij}}\}f^*_j(z) = u^*_i(z) \qquad \ldots \quad (2.2.7)$$

The variable $u^*_i(z)$ contains all the terms involving the initial conditions.

The set of equations (2.2.7) can also be written in matrix form

$$\underline{A}^*(z).\underline{f}^*(z) = \underline{u}^*(z) \qquad \ldots \quad (2.2.8)$$

The stability of the system will depend upon the form of the matrix $\underline{A}^*(z)$ and the final frequency could be deduced from the vector $\underline{u}^*(z)$.

2.3 Non-linear Systems

A number of non-linear systems have been proposed. Some are continuous systems, with non-linearities added to give weight to the larger phase differences; others are discontinuous systems. The first category has been investigated to some extent by Candy and Karnaugh[26] who suggest that there is little advantage to be gained from such systems. The second category has been extensively investigated by Duerdoth[6, 38], although some other work has been reported[9] by Mumford and Smith.

Both the Mumford and Smith system and the various Duerdoth systems use an oscillator that can produce an almost instantaneous phase step. This is achieved by switching from one frequency to another for a short time period. Such corrections are initiated only when the phase difference between two exchanges is such that the buffer stores on the circuits between them are likely to overflow. There is thus an alarm on every store to indicate that most of the store capacity has been used. This will persist until the spare capacity has been increased by an appreciable amount by means of the change in phase. There is thus hysteresis in the alarm circuit. Until an alarm is given, the phases drift within the limits set by the system designer. As soon as an alarm is acted upon, at least one oscillator changes phase. The store creating the demand for control action is then put into the centre of its available range, but other stores may well be put at an increased risk of overflow and create their own demands for control action. Thus the system is likely to spend a considerable time taking no action, followed by much activity to restore the required phase relationship.

In the system described by Mumford and Smith, control action to change an oscillator phase is taken only when a store at that exchange requires a correction. The system is thus single-ended. If all the delays are equal to integer multiples of one frame period, the stores will all be half full in the absence of phase errors. With phase errors, the stores at the two ends of a link will always have equal and opposite deviations from the centre of their ranges. Thus when one store demands action so will the store at the other end of the same link. The action taken will always be such as to bring the two oscillators nearer to a common phase, and thus the mean of the frequencies will tend to the average of the free-running frequencies. When the delays are not equal to integer multiples of one frame period, one store will require a correction before the other and the oscillator thus corrected will run at the frequency of the other (as in fig. 2.7). Provided that all the delays are above their proper values, or alternatively all below, the network will operate at either the highest or the lowest free-running frequency. If the delays are randomly distributed then the network cannot operate synchronously. Mumford and Smith avoid this problem by artificially biassing the delays in one direction such that natural variation of delays will not disturb the system. The system is still subject to the drawbacks of being essentially one with a master frequency to be followed by all the oscillators, and the buffer stores need to be of larger capacity.

Duerdoth avoids the difficulty by making all his systems double-ended. Even if a correction is demanded at one end of a link it is effective at both ends. It is thus feasible that the system can be made to work, but it is still to be shown that the operation is stable.

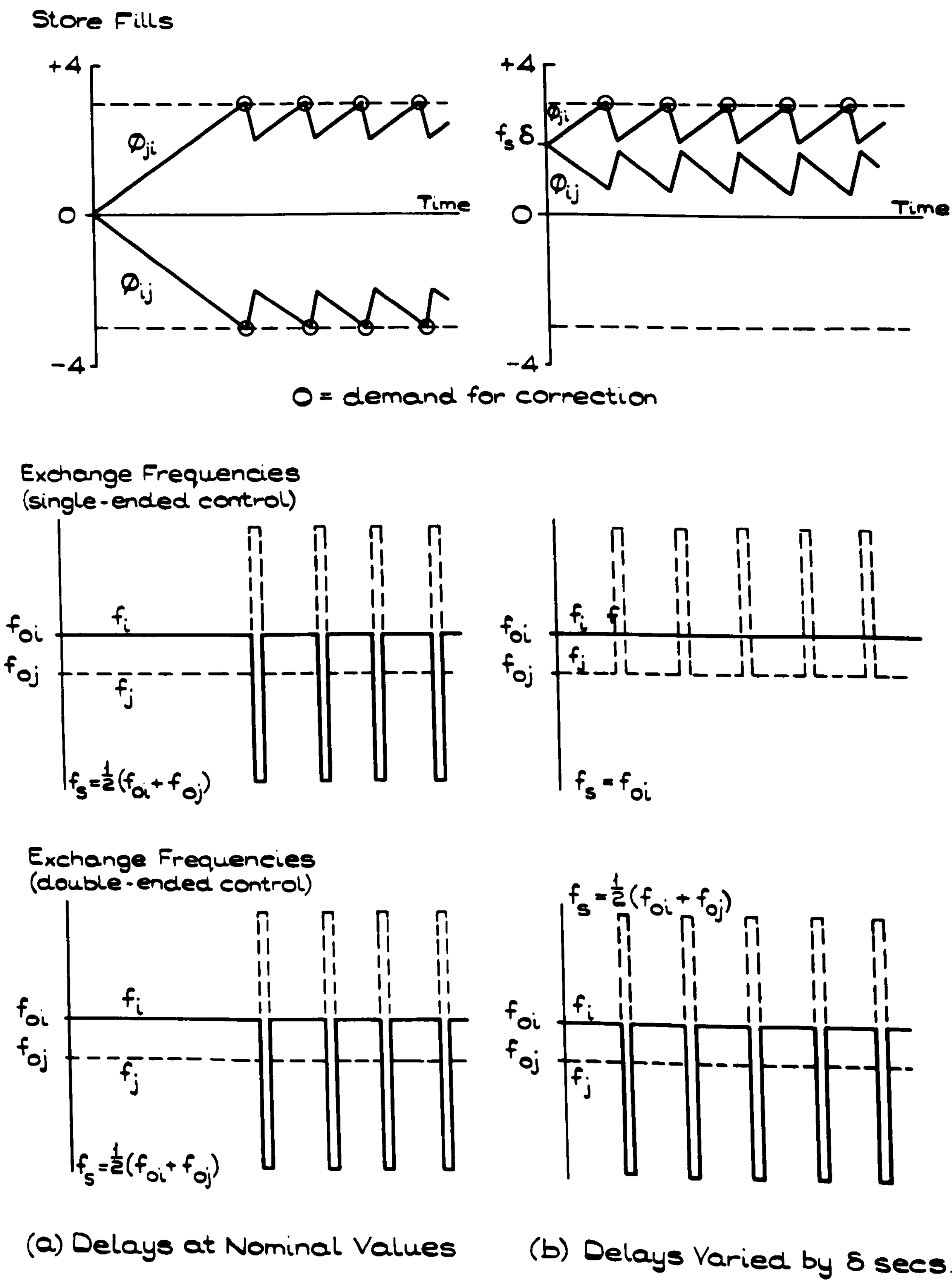


FIG. 2.7. OPERATION OF NON-LINEAR SYSTEMS (2 EXCHANGES).

The problem of wrong mode operation arises in the non-linear systems. This is because a correction cannot always be made in response to every demand for action. A demand will only show a phase error to exist and it is possible for conflicting demands to arise. As no knowledge of the frequency error is available, a majority decision logic is likely to aggravate the situation. Thus it has been decided to make a correction only when there is no conflicting demand and to suspend action if a conflicting demand should arise at any stage. There is of course the possibility of the correction being effective at the other end of the links concerned. However, if the phase differences are sufficiently large, all the exchanges in a particular area will receive conflicting demands. Instead of an equilibrium state being reached as in a linear system, blocking will occur. This is the condition corresponding to wrong mode operation. The recent proposals by Duerdoth[38] include means for the release of this condition.

3 STABILITY OF THE CONTROL SYSTEMS

The subject of stability has been studied by many workers, but there is as yet no general proof of stability of a generalised system. Such a system would include possible non-linearities, sampling whether single- or multi-rate, delays, filters and additional feedback paths as in the double-ended system. In the sections that follow, the stability of certain fairly general linear systems is discussed. The results include sufficient information to justify a large scale field trial of one of the methods of control described in the previous chapter.

For the linear systems, there is a matrix characteristic equation to be solved to yield the required stability condition. In all the cases considered here it has been possible to obtain a general sufficient condition without detailed knowledge of the size or shape of the network, and hence the entries of the matrix.

The equations for the general continuous systems are of the form

$$\{s\underline{I} - \underline{A}(s)\}\,\overline{\underline{f}} = \underline{c}$$

where the vector $\underline{c}$ is a function of the initial conditions and determines the system frequency. The characteristic equation determining stability is

$$\det.\left|s\underline{I} - \underline{A}(s)\right| = 0$$

For stability the roots in $s = \sigma + j\omega$ must have negative real parts ($\sigma < 0$). The subsequent sections will determine the conditions for this to be so.

A sampled system has equations as a function of $z = e^{+sT_s}$. These are of the form

$$\{\underline{A}^*(z)\}\ \underline{f}^*(z) = u^*(z)$$

For stability of this type of system the roots in z of the characteristic equation

$$\det.|\underline{A}^*(z)| = 0$$

must have modulus less than unity. The conditions for this to be so are less easy to determine, but the results obtained so far will be given.

3.1 Stability of single-ended systems

3.1.1 Systems without line delays or filters

One of the first results obtained was for this simple model[11]. In this special case, the matrix $\underline{A}(s) = \underline{A}(0) = k\underline{A}$, where $\underline{A}$ is a constant matrix and $k_{ij} = k$ for all i, j. The roots of the characterstic equation are then k times the eigenvalues of the connection matrix $\underline{A}$. At the suggestion of Mr. Parks, the Gershgorin Circle Theorem[10] was applied to the problem of locating these eigenvalues.

The Gershgorin theorem states that the eigenvalues of a complex matrix lie in that part of the complex plane which is the union of the set of circles for which the coordinates of the centres are the diagonal elements of the matrix and the radii are the sum of the moduli of the off-diagpnal elements of the same row. Columns may be used instead of rows.

In this case the matrix $\underline{A}$ is symmetric and has integer elements. Thus the eigenvalues are real. The diagonal elements are $a_{ii} = -m_i$, where $m_i \leq (n - 1)$ is the number of exchanges directly connected to the ith exchange. Of the $(n - 1)$ off-diagonal elements there will be m_i with value +1; the others will be zero. Thus all the circles have their centres in the left-hand half-plane and pass through the origin. No roots can have positive real parts, so the system is stable.

One of the roots of the characteristic equation is at $s = 0$. This is also true of systems with delays or filters. The eigenvector of the matrix $\underline{A}$ corresponding to this eigenvalue is $\{1, 1, 1, \ldots, 1\}$. This corresponds to the condition that all the frequencies are equal in the steady state. The value of the steady state frequency will be denoted by f_s.

It is possible to remove the zero root (which corresponds to a redundant equation) by subtraction of one of the system equations (2.1.5) from all the others. If this is the nth equation and the nth exchange is connected to all the others, it is possible to derive a new set of Gershgorin circles passing through the point $(-n, 0j)$ and with centres at $(-m_i - 1, 0j)$. Thus a lower bound on all the roots is $-n$; an upper bound for the least negative non-zero root is also possible in well connected systems. This will be made use of when considering transient behaviour.

The fact that this system is stable for all values of gain k, whatever the network configuration, is remarkable. It is possible to extend this at once to the case of different values of gain k_{ij} on each input since the sum of the elements of any row of $\underline{A}(0)$ remains zero and the diagonal element is negative.

This result was incorporated in the writer's M.Sc. dissertation in May 1967, having been published in December 1966[11]. It was republished, without reference, by West in October 1967[20].

3.1.2 Effect of line delays

In this early work, an attempt to consider line delays had also been made. This required the inclusion of artificial delays as an analytical convenience. The system thus looked more like the double-ended system, and it was shown that there was an upper bound on the product of the common gain k and the delay d applied to all parts of the system. This method was somewhat unsatisfactory.

In his paper[20], West quoted the paper of Gersho and Karafin[12] and derived a stability condition based on one given in that paper. This appears to be the unpublished condition by Beneš, mentioned by Karnaugh[13].

This condition relies on an application of the Gershgorin theorem, known as 'diagonal dominance'. If, for all $\sigma \geq 0$, $s \neq 0$, the matrix $\{s\underline{I} - \underline{A}(s)\}$ has its elements such that the modulus of the diagonal term is greater than the sum of the moduli of the off-diagonal terms of the same row, then no Gershgorin circle includes the origin. The matrix is thus non-singular for all s in the right-hand half-plane, and stability is assured.

The diagonal dominance condition that

$$|s - a_{ii}(s)| > \sum_{\substack{j=1 \\ \neq i}}^{n} |a_{ij}(s)| \qquad \ldots \quad (3.1.1)$$

can be rewritten as

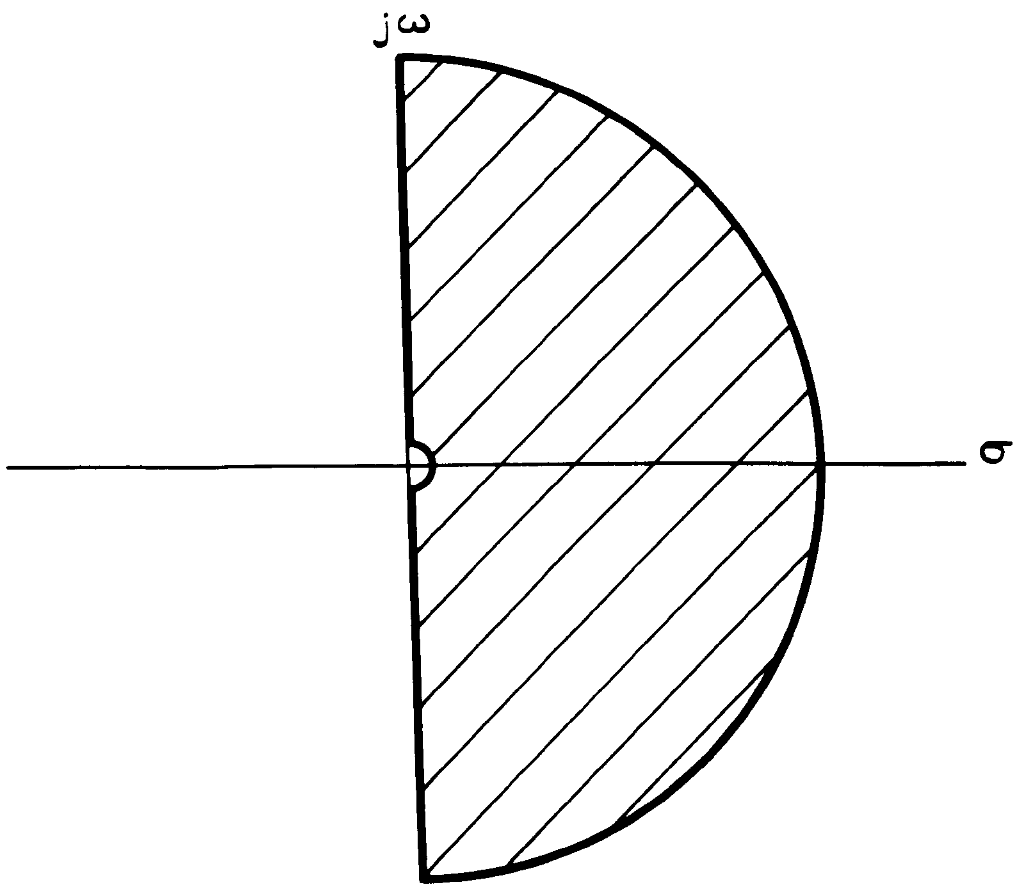
$$\frac{\sum\limits_{\substack{j=1 \\ \neq i}}^{n} |a_{ij}(s)|}{|s - a_{ii}(s)|} < 1$$

for all non-zero s on or to the right of the imaginary axis.
In this Thesis the filters will be such that this expression may be manipulated to

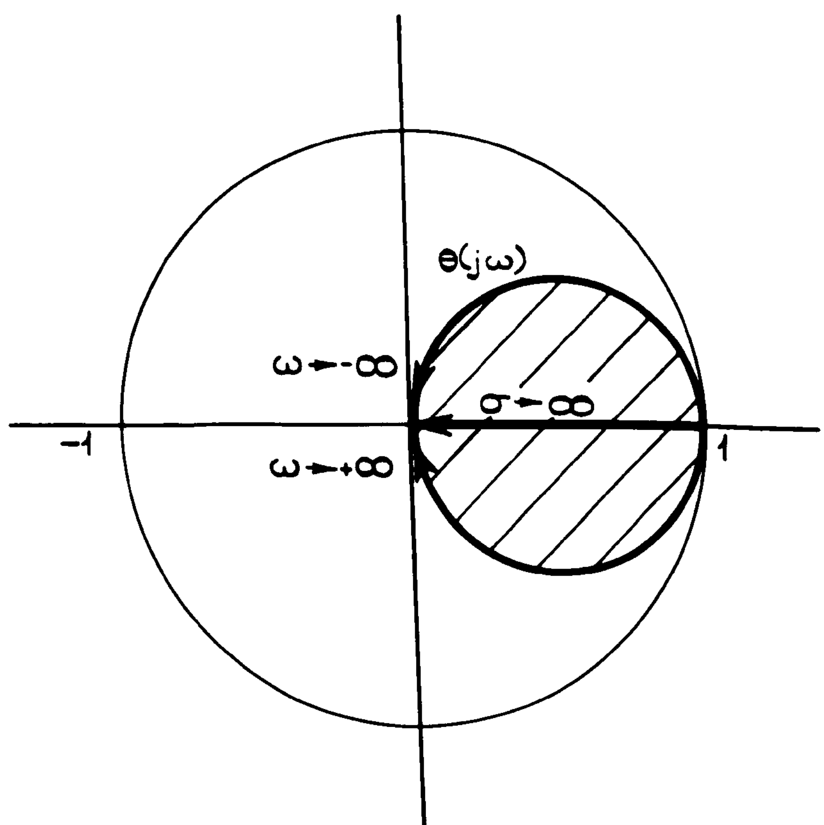
$$\left|\theta(s)\right| = \left|\frac{\phi(s)}{\psi(s)}\right| < 1 \qquad \ldots \quad (3.1.2)$$

where $\psi(s) = \dfrac{s}{\sum\limits_{\substack{j=1 \\ \neq i}}^{n} k_{ij}} + \chi(s)$ and $\chi(s)$ is independent of k_{ij}.

It will be possible to restrict the examination of $\theta(s)$ to $s = j\omega$. If it can be shown that $|\theta(j\omega)| < 1$, then the locus of $\theta(j\omega)$ lies within the unit circle. For $\sigma > 0$, $\theta(\sigma)$ is real in the range $0 \leq \theta(\sigma) < 1$, where $\theta(\infty) = 0$. Furthermore, as $\omega \to \infty$, $\theta(j\omega) \to -0j$ from below. In general, the right-hand half of the s-plane maps via $\theta(s)$ into the inside of the locus of $\theta(j\omega)$. This is illustrated in fig. *3.1*. As there are no poles of $\theta(s)$ in the right-hand half-plane, the maximum modulus of $\theta(s)$ is to be found on $\theta(j\omega)$. Thus the satisfaction of the diagonal dominance condition for $s = j\omega, \neq 0$ implies the satisfaction of the condition for the whole of the region $s \neq 0$, $\sigma > 0$.



(a) $s = \sigma + j\omega$



(b) $\theta(s)$

FIG. 3.1. MAPPING OF S-PLANE.

For the simple model considered here, when $s = j\omega$ and $\omega \neq 0$ the condition that

$$\left| j\omega + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \right| > \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \left| e^{-j\omega d_{ij}} \right| \qquad \ldots \quad (3.1.3)$$

is clearly satisfied for all values of k_{ij}.

West[20] derived this result with the restriction that all $k_{ij} = k.a_{ij}$ and used a less general method to show that the maximum modulus theorem holds.

3.1.3 Effect of filters

In this, the general case of a linear single-ended system, the diagonal dominance condition for system stability is

$$\left| s + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} F_{ij}(s) \right| > \sum_{\substack{j=1 \\ \neq i}}^{n} \left| k_{ij} F_{ij}(s) G_{ij}(s) e^{-sd_{ij}} \right| \qquad \ldots \quad (3.1.4)$$

for each value of i, where $s \neq 0$ and $\omega \geq 0$.

The delay term, which appears only on the right-hand side, can be neglected as it does not affect the modulus for $s = j\omega$. It will be assumed that, for all i, j, $G_{ij}(s) = G_i(s)$. The stability condition may now be written as

$$\left| G_i(j\omega) \right| . \left| \frac{\sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} . F_{ij}(j\omega)}{j\omega + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} . F_{ij}(j\omega)} \right| < 1 \qquad \ldots \quad (3.1.5)$$

for non-zero ω.

This may be satisfied in one of three ways. If both the moduli are less than unity the stability is assured. Alternatively one of the moduli could be greater than unity at some ω, the other being sufficiently less than unity for the product to be less than unity. In practice there is no transient overshoot from the recovery filters, which implies a stronger condition than $|G_{ij}(j\omega)| < 1$. Thus a sufficient condition is for the remaining modulus to be less than unity for all ω.

By considering the $F_{ij}(s)$ to represent a first-order low-pass filter in the feedback loop, such that $F_{ij}(s) = \frac{1}{1 + sT_{Fij}}$, it is clear that the second term on the left-hand side of expr. (*3.1.5*) is the modulus of a second order function. This may have a resonance. In general, the order of this term is one greater than the order of the filter within the feedback loop.

When all the filters at the ith exchange are identical, namely when $T_{Fij} = T_{Fi}$ for all i, j, expr. (*3.1.5*) may be simplified to the condition that

$$\left|G_i(j\omega)\right| \cdot \left|\frac{1}{1 + \frac{j\omega}{K}(1 + j\omega T_{Fi})}\right| < 1 \text{ for } \omega \neq 0 \qquad \ldots \quad (3.1.6)$$

where $K = \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}$.

The sufficient stability condition, with both transfer functions non-resonant, is

$$\left|G_i(j\omega)\right| < 1 \quad \text{and} \quad T_{Fi}\left\{\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\right\} < \tfrac{1}{2} \qquad \ldots \quad (3.1.7)$$

The condition of Hills[30] is equivalent to this for the values of gain he uses, namely that all $k_{ij} = k$. West[20], however, obtains a condition that requires a smaller time constant-gain sum product. He appears to have used the condition for no transient overshoot which is more restrictive than the condition for non-resonance.

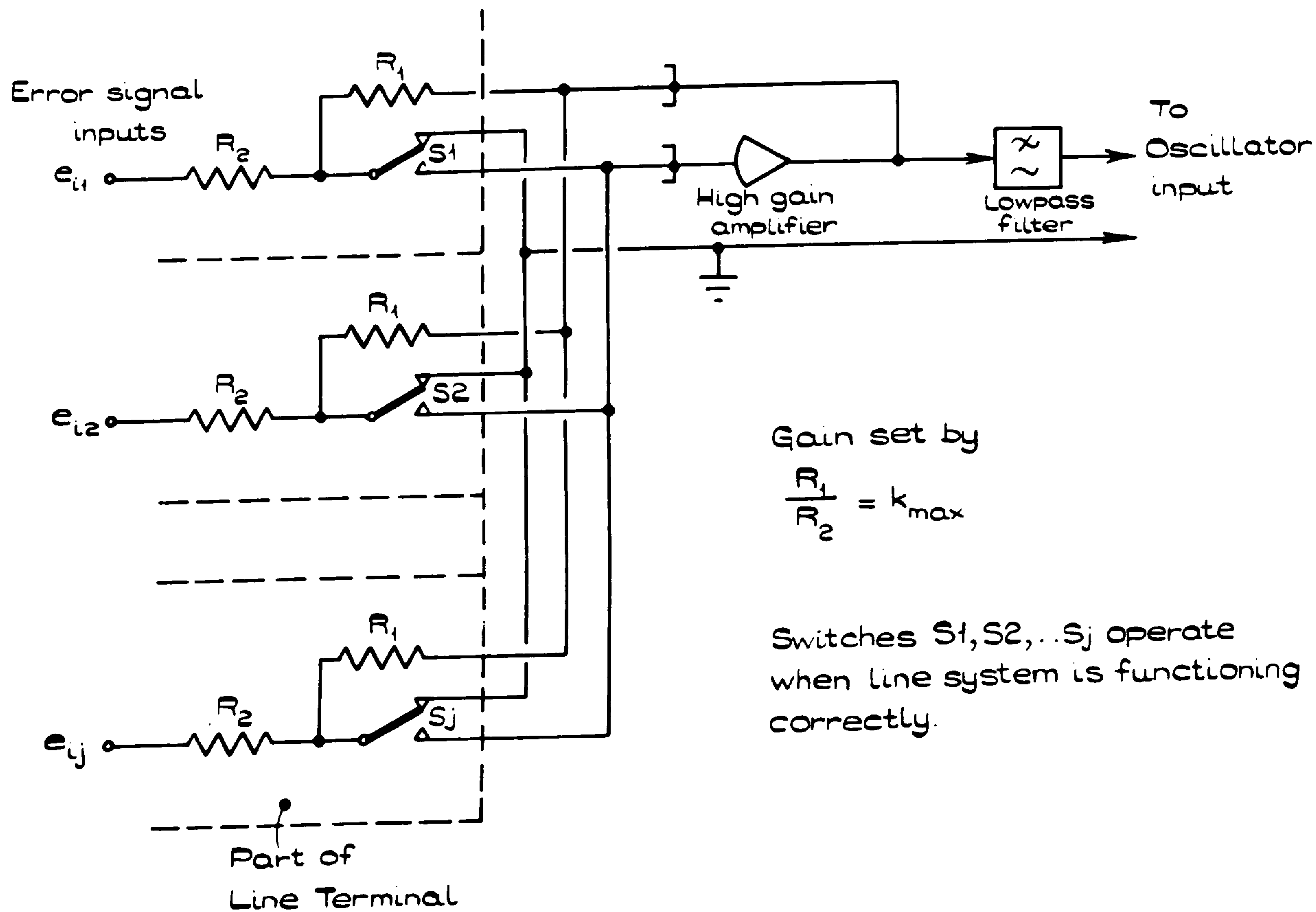
Using standard notation, for the second order filter with transfer function

$$F(s) = \frac{1}{1 + 2\xi\frac{s}{\omega_0} + \left\{\frac{s}{\omega_0}\right\}^2}$$

the condition for non-resonance is $\xi > \frac{1}{\sqrt{2}}$ and the condition for no transient overshoot is $\xi > 1$. This accounts for the factor of 2 in the comparison of West's result.

The stability of the single-ended system is thus dependent on the time constants of the filters and the gain applied at each exchange. Provided that at each exchange the gains are compensated according to the number of exchanges connected, the network stability can be assured.

It will be seen later that high gains reduce the recovery time of the system from a transient overshoot. It is thus desirable to maintain the gains at the highest possible level, subject to the stability condition being satisfied. One possible solution is to make all the gains $k_{ij} = k_i.a_{ij}$ where $k_i.m_i = k_{max}$. The connection



Gain set by

$$\frac{R_1}{R_2} = k_{max}$$

Switches S1, S2, .. Sj operate when line system is functioning correctly.

FIG. 3.2. AUTOMATIC GAIN CONTROL

factors $a_{ij} = a_{ji}$ and are unity when the ith and jth exchanges are interconnected and zero otherwise. The sum of the connection factors

$$\sum_{\substack{j=1 \\ \neq i}}^{n} a_{ij} = m_i$$

and k_{max} is the value of gain allowed by stability considerations as a maximum throughout the network. A device[40] that will maintain this is shown in fig. 3.2. The switches shown are operated when the incoming line terminal is correctly receiving synchronisation pulses from the distant exchange. Devices to check these pulses are part of every system design.

3.2 Stability of double-ended systems

These systems are more complicated than the single-ended systems because of the provision of the data link. Although the data link will form part of the p.c.m. system, it is assumed that the link delay will be different (and usually longer) than the line delay in the same direction. This allows some consideration of different methods of providing this link. In practice, the signals on the link will be sampled and this leads to the systems discussed is section 3.4.

It will be seen that the data link delay term appears on the diagonal of the system matrix. This will introduce delay as an important factor in the determination of stability. Apart from the simplest cases, the diagonal dominance method will be used to obtain a stability condition. The systems parameters must satisfy the condition that for all $\sigma \geq 0$, $s \neq 0$ and for each i

$$\left| s + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \left\{ F_{ij}(s) + G_{ji}(s) . H_{ij}(s) . e^{-s(\tau_{ij} + d_{ji})} \right\} \right|$$

$$> \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \left\{ |F_{ij}(s) G_{ij}(s)| + |H_{ij}(s)| \right\}$$

$$> \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \left| F_{ij}(s) . G_{ij}(s) . e^{-sd_{ij}} + H_{ij}(s) . e^{-s\tau_{ij}} \right| \quad \ldots \quad (3.2.1)$$

3.2.1 No delays or filters

In this simple case, the only difference from the single-ended system is that the effect of the gains is doubled. Thus, as in section 3.1.1, it can be shown that the system is unconditionally stable.

3.2.2 Effect of delays

The first paper published by Parks and Miller[11] included a result for this case by means of artificially increasing the delays. As expected, the following conditions with normal delays allow the use of larger gains than suggested by the early stability criterion.

The result published by Miller[36], quoted earlier without proof[21], concerns only the case where the system is balanced, that is where all $G_{ij}(0) = 1$ and $F_{ij}(0) = H_{ij}(0)$. Candy and Karnaugh[26] give a condition for an unbalanced system, which assumes that all delays are equal and that all exchanges apply the same gain to each input signal. The result which follows removes these restrictions, and indicates the connection between the two published results.

The filter terms in expr. (*3.2.1*) will be removed, but, to allow for the gain to be different in the forward and backward paths, it is assumed that, for all s,

$$k_{ij}.F_{ij}(s) = \alpha_{ij}, \quad k_{ij}.H_{ij}(s) = \beta_{ij} \quad \text{and} \quad G_{ij}(s) = 1.$$

Expr. (*3.2.1*) now becomes

$$\left| s + \sum_{\substack{j=1 \\ \neq i}}^{n} \alpha_{ij} + \sum_{\substack{j=1 \\ \neq i}}^{n} \beta_{ij} e^{-s(\tau_{ij}+d_{ji})} \right| > \sum_{\substack{j=1 \\ \neq i}}^{n} (\alpha_{ij} + \beta_{ij}) \qquad \dots \quad (3.2.2)$$

It can be seen from fig. 3.3 that the critical values of $s = j\omega$ lie in the region $0 < \omega D_{im} < \pi/2$, where

$D_{im} = (\tau_{ij} + d_{ji})_{\max j}$, $j = 1, 2, \dots n$.

Then

$$\left| j\omega + \sum_{\substack{j=1 \\ \neq i}}^{n} \alpha_{ij} + \sum_{\substack{j=1 \\ \neq i}}^{n} \beta_{ij} e^{-j\omega(\tau_{ij}+d_{ji})} \right|$$

$$> \left| j\omega + \sum_{\substack{j=1 \\ \neq i}}^{n} \alpha_{ij} + e^{-j\omega D_{im}} \sum_{\substack{j=1 \\ \neq i}}^{n} \beta_{ij} \right| \qquad \dots \quad (3.2.3)$$

Where $\sum_{\substack{j=1 \\ \neq i}}^{n} \alpha_{ij} = \alpha_i$, $\sum_{\substack{j=1 \\ \neq i}}^{n} \beta_{ij} = \beta_i$ the condition becomes

$$|j\omega + \alpha_i + \beta_i e^{-j\omega D_{im}}| > \alpha_i + \beta_i \qquad \dots \quad (3.2.4)$$

This gives the vector diagram of fig. 3.3. From this it may be seen that

$$2\alpha_i\beta_i(1 - \cos \omega D_{im}) + 2\omega\beta_i \sin \omega D_{im} - \omega^2 < 0 \qquad \dots \quad (3.2.5)$$

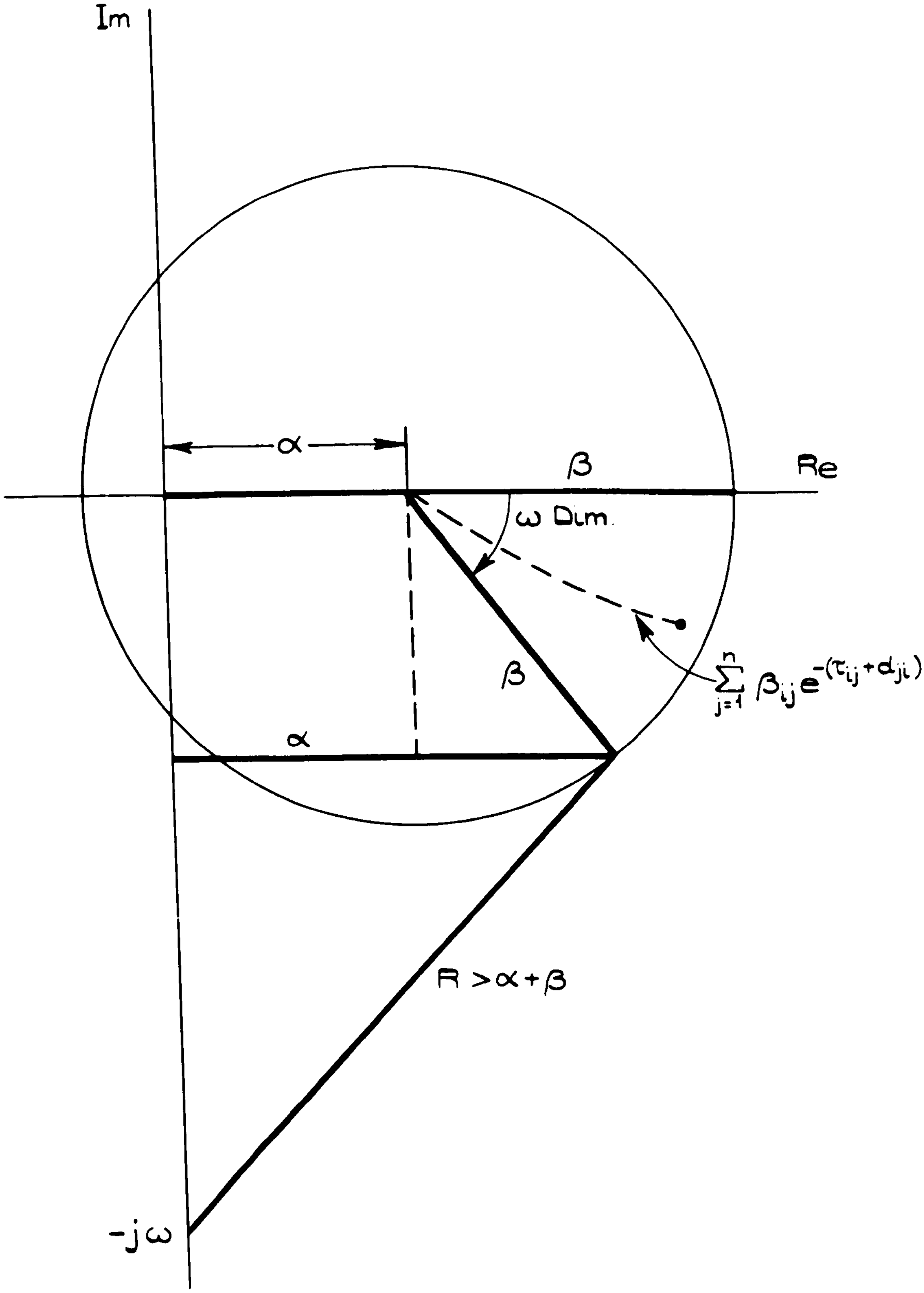
Letting $\omega D_{im} \to 0$ clearly gives the limiting condition that

$$2\alpha_i\beta_i(1 - 1 + \tfrac{1}{2}\omega^2 D_{im}^2) + 2\omega\beta_i(\omega D_{im}) - \omega^2 < 0$$

Since $\omega \neq 0$

$$2\alpha_i\beta_i\left(\frac{D_{im}^2}{2}\right) + 2\beta_i D_{im} - 1 < 0$$

whence $\quad \beta_i D_{im}(2 + \alpha_i D_{im}) < 1 \qquad \dots \quad (3.2.6)$



Im
Re
α
β
ω Dim.
β
$\sum_{j=1}^{n} \beta_{ij} e^{-(\tau_{ij}+d_{ji})}$
α
$R > \alpha + \beta$
$-j\omega$

FIG. 3.3

This condition is clarified by reference to fig. 3.4. The parameters must be such that the working point lies in the first quadrant under the rectangular hyperbola.

This figure illustrates three special cases. At point A, $\alpha_i = \beta_i = \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}$, which is the condition for the system to be balanced. The stability condition is

$$(d_{ji} + \tau_{ij})_{\max\, j},\ j=1,\ 2,\ \ldots\ n \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} < \sqrt{2} - 1 \qquad \ldots \quad (3.2.7)$$

which has already been published[20, 36].

The previously unpublished condition of Brilliant, quoted without proof by Candy and Karnaugh[26], is that for all α_i, β_i, $(\alpha_i + \beta_i)\, D_{im} < \frac{1}{2}$. This restricts the working point to the origin side of the straight line passing through B.

When $\beta_i = 0$, the condition becomes that of the single-ended system in which there is no limit to the gain-delay product $\alpha_i D_{im}$.

It should be noted that the stability condition refers to the maximum loop delay between an exchange i and any of those to which it is connected, rather than requiring that all the delays be equal, as in the condition of Brilliant. Thus the condition can be checked at each exchange, without reference to other exchanges. The equipment design shown in fig. 3.2 is again suitable.

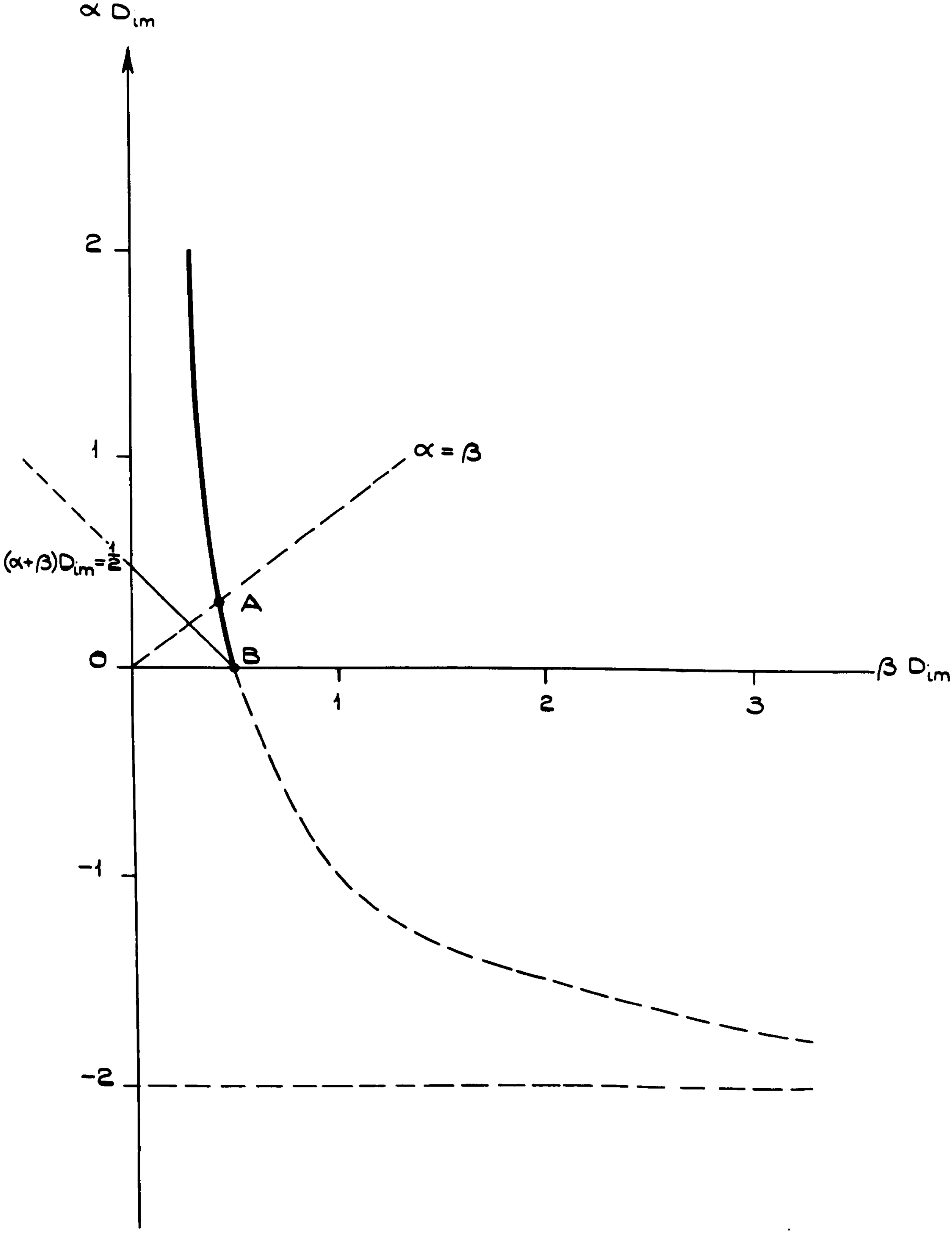


FIG. 3.4

3.2.3 Fully interconnected system with delays but no filters

For this special case we will assume that all the delays are equal to d, and the gains equal to k. It is now possible to factorise the characteristic equation $|s\underline{I} - \underline{A}(s)| = 0$ to give

$$\{s + (n - 1)k(1 + e^{-2sd} - 2e^{-sd})\}.$$
$$\{s + (n - 1)k(1 + e^{-2sd} - 2e^{-sd}) + 2nk\, e^{-sd}\}^{n-1} = 0 \qquad \ldots \quad (3.2.8)$$

If either of these factors is zero for a value of s in the right-hand half of the s-plane then the system is unstable.

The first factor is, as usual, zero for s = 0. The remainder can be rearranged to

$$\frac{(n - 1)k}{s}\left\{1 - 2e^{-sd} + e^{-2sd}\right\} = -1 \quad \text{for } s \neq 0 \qquad \ldots \quad (3.2.9)$$

The critical case is given by $s = j\omega$, for which $\frac{(n - 1)k}{j\omega}$ is imaginary. Thus the left hand side is real when $(1 - e^{-j\omega d})^2$ is imaginary. This occurs when $\omega d = \frac{1}{2}\pi(1 + 2m)$. The even values of m give the left-hand side of expr. (3.2.9) a positive value, so the odd values of m are critical. The resulting condition is $(n - 1)kd < 3\pi/4$.

The same method applied to the second factor again gives $\omega d = \frac{1}{2}\pi(1 + 2m)$ for the left-hand side to be real. Here it is the even values of m that are critical. The resultant condition is that $kd < (\pi/4)$.

It is thus clear that the system is stable if, for $n \leq 4$,

$kd < \frac{\pi}{4}$ and for $n \geq 4$ if $(n - 1)kd < \frac{3\pi}{4}$.

Yamato et al.[25] give only the first condition and omit the observation that it only applies up to $n = 4$. The above correction was published by Parks and Miller[29] and republished later with a corrected figure for the stability region[37]. This condition is both necessary and sufficient.

It is interesting to note that the sufficient condition of expr. (*3.2.7*) gives $(n - 1)kd < \frac{1}{2}(\sqrt{2} - 1) = 0{\cdot}207$ for all n, against the minimum value of 0·785 from the necessary and sufficient condition when $n = 2$. As the sufficient condition applies to network configurations less than fully interconnected, the difference is acceptable.

3.2.4 Effect of filters

In the double-ended system the filter time constants as well as the delays will affect the stability. In this section the filters will be considered alone, and the next section will show the combined effect.

It will be assumed here that all the filters of each type at a single exchange are identical, and $G_{ij}(s) = G(s)$ for all i, j. Thus expr. (*3.2.1*) reduces to

$$\left| s + \{F_i(s) + G(s)H_i(s)\} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \right| > \left| F_i(s)G(s) + H_i(s) \right| \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \qquad \ldots (3.2.10)$$

With $G(s) = 1$ and $s = j\omega$, it can be seen from the vector diagram of fig. 3.5 that the condition holds if

$$\frac{j\omega}{\sum\limits_{\substack{j=1\\ \neq i}}^{n} k_{ij}} > -2\,\mathrm{Im}\{F_i(j\omega) + H_i(j\omega)\}$$

Now it is clear that, for all ω, $-\mathrm{Im}\{F_i(j\omega) + H_i(j\omega)\} \leq \omega(T_{Fi} + T_{Hi})$ where the filters are both of the first order low pass type. Thus a sufficient condition for the stability of the system is that

$$(T_{Fi} + T_{Hi}) \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij} < \tfrac{1}{2} \qquad \ldots (3.2.11)$$

Consideration of the filter G(s) requires evaluation of the moduli. As $\omega \to 0$ it can be seen that the sufficient condition becomes

$$(T_{Fi} + T_G + T_{Hi}) \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij} < \sqrt{2} - 1 \qquad \ldots (3.2.12)$$

By differentiation, as in the next section, it can be shown that if the diagonal dominance condition is satisfied for small ω it is also satisfied for larger ω. In the special case where $T_{Hi} \leq T_{Fi}$ (which is unlikely in practice) the right-hand side of expr. (3.2.12) could be increased to ½.

3.2.5 Effect of filters and delays

The first case to be discussed in this section is for $G_{ij}(s) = G(s) = 1$. The filter G(s) will be included later. It will

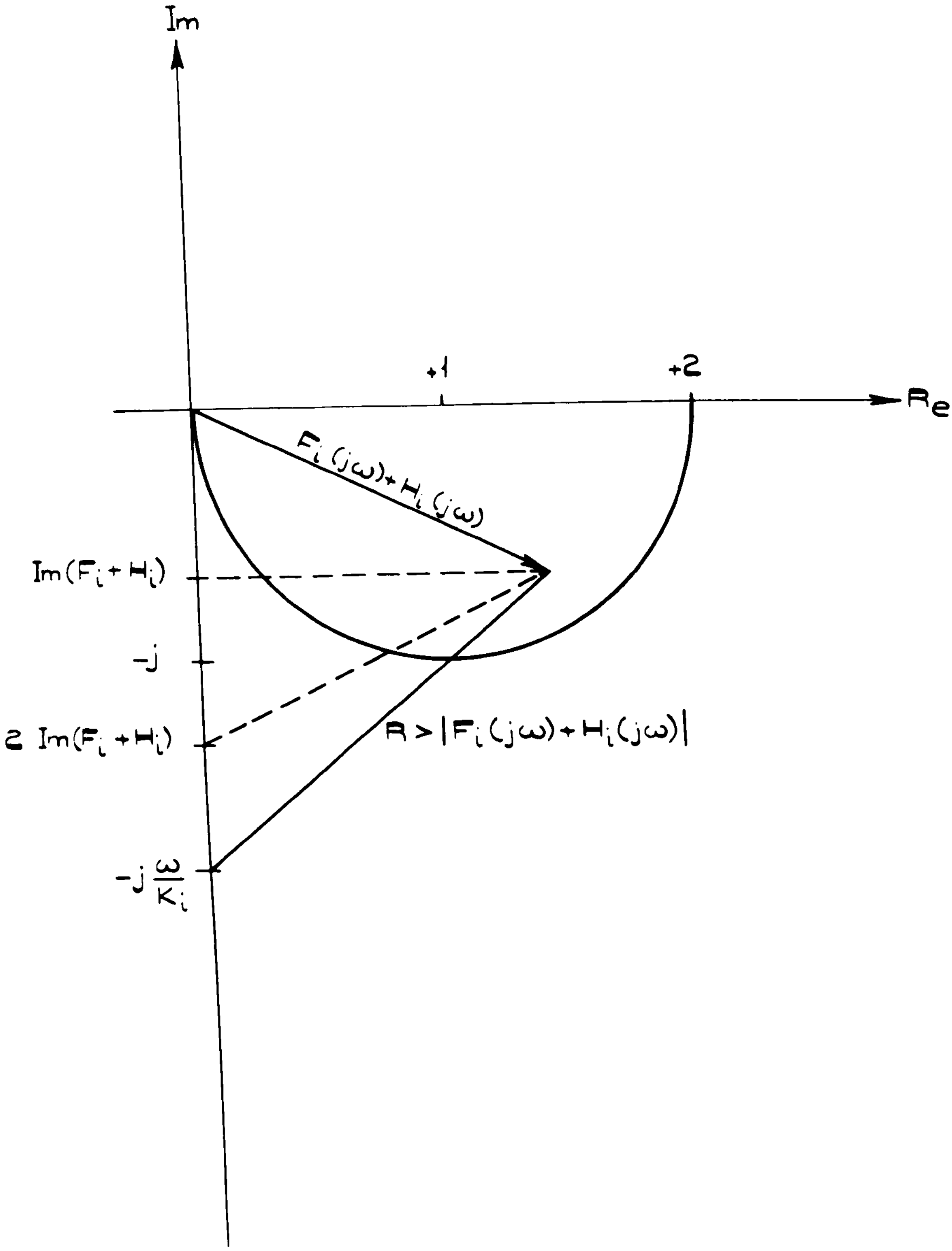


FIG. 3.5. STABILITY DIAGRAM: DOUBLE-ENDED SYSTEM WITH FILTERS.

be seen that the stability condition is that of expr. (*3.2.12*) with a delay term added to the bracket on the left-hand side.

As in the earlier work it is only necessary to consider $s = j\omega$ to establish the complete stability condition. For $\omega D_{iM} < \frac{\pi}{2}$ it is possible to put D_{iM} in place of $(\tau_{ij} + d_{ji})$ where $D_{iM} = (\tau_{ij} + d_{ji})_{\max\, j}$; this reduces the value of the left-hand side of expr. (*3.2.1*).

If $F_{ij}(s) = F_i(s)$, $H_{ij}(s) = H_i(s)$ and $K_i = \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}$ this expression may be rewritten as

$$\left|\frac{j\omega}{K_i} + F_i(j\omega) + H_i(j\omega).e^{-j\omega D_{iM}}\right| > \left|F_i(j\omega)\right| + \left|H_i(j\omega)\right| \qquad \ldots (3.2.13)$$

for each value of i and $\omega \neq 0$.

Simple first-order low-pass filters will be assumed, thus $F_i(s) = 1/(1 + sT_{Fi})$ and T_{Hi} is similarly defined. Inspection of fig. *3.6* shows that for $\omega > 3{\cdot}5\ K_i$ the condition (*3.2.13*) is always satisfied. It can now be seen that when $K_i D_{iM} < \pi/7 = 0{\cdot}4489$, expr. (*3.2.1*) is satisfied for $\omega > \pi/2D_{iM}$. It is thus only necessary to examine values of ω up to this limit, and it will be found that a greater restriction on values of $K_i D_{iM}$ exists within this range of ω.

The right-hand side of this expression (*3.2.13*) decreases with increasing ω. By differentiation of the left-hand side with respect to ω it is possible to show that this increases with ω. The limit as $\omega \to 0$ is therefore critical.

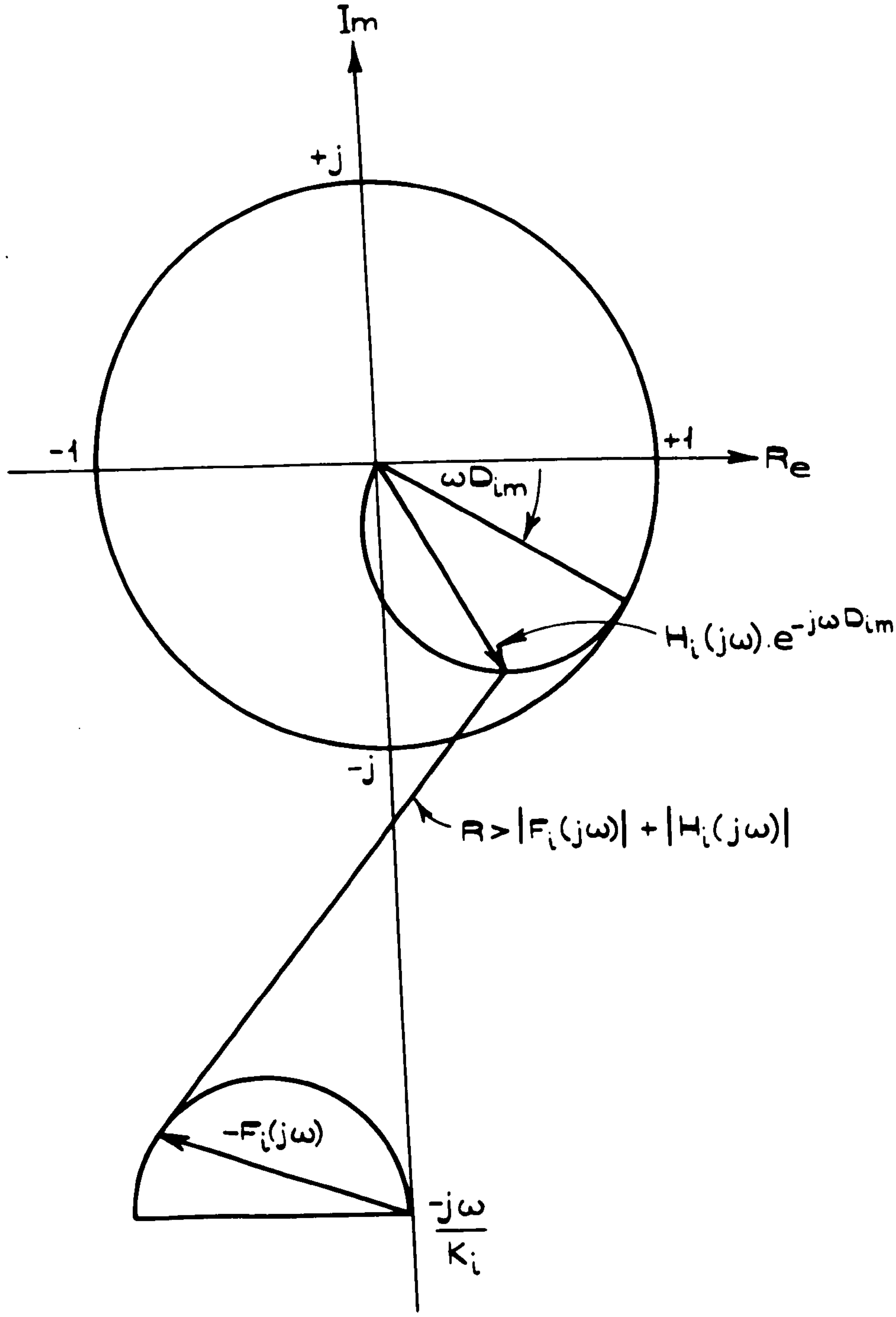


FIG. 3.6. STABILITY DIAGRAM: DOUBLE-ENDED SYSTEM WITH DELAYS AND FILTERS.

For $\omega \to 0$, expr. (3.2.13) becomes

$$\Big\{4 - 4\omega^2(T_{Fi}^2 + T_{Hi}^2 + \tfrac{1}{2}D_{iM}^2 + T_{Hi}D_{iM}) + \text{order}(\omega^4)$$

$$+ \omega^2(\frac{1}{K_i} - T_{Fi} - T_{Hi} - D_{iM})\Big\}^{\frac{1}{2}}$$

$$> \Big\{1 + \omega^2 T_{Fi}^2\Big\}^{-\frac{1}{2}} + \Big\{1 + \omega^2 T_{Fi}^2\Big\}^{-\frac{1}{2}} \qquad \ldots (3.2.14)$$

Thus

$$2\Big\{1 - \tfrac{1}{2}\omega^2(T_{Fi}^2 + T_{Hi}^2 + \tfrac{1}{2}D_{iM}^2 + T_{Hi}D_{iM})$$

$$+ \tfrac{1}{8}\omega^2(\frac{1}{K_i} - T_{Fi} - T_{Hi} - D_{iM})^2 + O(\omega^4)\Big\}$$

$$> 2\Big\{1 - \tfrac{1}{2}\omega^2(T_{Fi}^2 + T_{Hi}^2) + O(\omega^4)\Big\} \qquad \ldots (3.2.15)$$

After cancellation of terms and division by $\frac{1}{8}\omega^2$, expr. (3.2.15) becomes

$$-2(T_{Fi}^2 + T_{Hi}^2 + D_{iM}^2 + 2T_{Hi}D_{iM}) + \Big\{\frac{1}{K_i} - (T_{Fi} + T_{Hi} + D_{iM})\Big\}^2 > 0$$

By subtracting terms from both sides and putting $\lambda = (T_{Fi} + T_{Hi} + D_{iM})$ this yields

$$1 - 2\lambda K_i - \lambda^2 K_i^{\,2} > 0 > -2T_{Fi}(T_{Hi} + D_{iM})$$

A sufficient condition is that

$$\lambda K_i < \sqrt{2} - 1 \qquad \ldots (3.2.16)$$

A sharper condition could clearly be deduced, but the additional complexity is not regarded as justified by the small differences involved.

The more complicated expression involving the filter G(s) yields the general result that the double-ended system is stable if, for each value of i,

$$(D_{iM} + T_{Fi} + T_G + T_{Hi}) \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} < \sqrt{2} - 1 \qquad \text{... (3.2.17)}$$

It is now necessary to prove that the limiting case of $\omega \to 0$ is in fact the point at which the modulus defined in eqn. (*3.1.2*) is a maximum. The lengths F and H are defined by $F = |F_{ij}(j\omega)| = 1/\sqrt{(1 + \omega^2 T_{Fi}^2)}$ and $H = 1/\sqrt{(1 + \omega^2 T_{Fi}^2)}$. The length M is defined by

$$M^2 = (F\cos\theta + H\cos\phi)^2 + \left(\frac{\omega}{K_i} - F\sin\theta - H\sin\phi\right)^2$$

(see also fig. *3.6*)

It is required that $Y = \dfrac{d(M^2)}{d\omega} \geq 0$ for $\omega D_{iM} < \frac{1}{2}\pi$ with the other parameters limited by expr. (*3.2.17*).

Differentiating, $\dfrac{dF}{d\omega} = -2\omega T_{Fi}^2 F^3$ and similarly $\dfrac{dH}{d\omega} = -2\omega T_{Hi}^2 H^3$.
The angles θ and ϕ are defined by

$$\tan\theta = \omega T_{Fi} \quad \text{and} \quad \tan(\phi - \omega D_{iM}) = \omega T_{Hi}.$$

Differentiating, $\dfrac{d\theta}{d\omega} = T_{Fi}\cos^2\theta$ and $\dfrac{d\phi}{d\omega} = D_{iM} + T_{Hi}\cos^2(\phi - \omega D_{iM})$

As $\omega \to 0$, $\theta \to \omega T_{Fi}$ and $\phi \to \omega(T_{Hi} + D_{iM})$

Now it can be seen that

$$\begin{aligned} Y = {} & 2(F\cos\theta + H\cos\phi) \\ & \Big\{-F\sin\theta . T_{Fi}\cos^2\theta + \cos\theta . (-2\omega T_{Fi}^2 F^3) \\ & \quad - H\sin\phi\{D_{iM} + T_{Hi}\cos^2(\phi - \omega D_{iM})\} - 2\omega T_{Hi}^2 H^3 \cos\phi\Big\} \\ & + 2\left(\frac{\omega}{k_i} - F\sin\theta - H\sin\phi\right) \\ & \Big\{\frac{1}{k_i} - F\cos\theta . T_{Fi}\cos^2\theta - H\cos\phi . \left[D_{iM} + T_{Hi}\cos^2(\phi - \omega D_{iM})\right] \\ & \quad + 2\omega T_{Fi}^2 F^3 \sin\theta + 2\omega T_{Hi}^2 H^3 \sin\phi\Big\} \end{aligned} \qquad \text{... (3.2.18)}$$

$$Y \to a\omega + \text{order}(\omega^2) + R(\omega^3) \text{ as } \omega \to 0 \text{ where } a > 0$$

As ω increases from 0 each of the positive terms in the above expression (*3.2.18*) increase, and the negative terms decrease, from the values which give the coefficient a.

There are, however, some terms in Y which have order ω^3 and therefore make no contribution to a. These terms are the remainder $R(\omega^3)$ where

$$R(\omega^3) = \left\{\frac{4\omega^2}{k_i} - 4\omega F \sin\theta - 4\omega H \sin\phi\right\}\left\{T_{Fi}F^3 \sin\theta + T_{Hi}H^3 \sin\phi\right\}$$

If this term is positive for $\omega > 0$, then $Y > 0$ for all ω and $\omega \to 0$ is the critical case.

The second term is positive. Thus the required condition is that

$$\frac{d}{d\omega}\left\{\frac{\omega}{k_i} - F \sin\theta - H \sin\phi\right\} > 0$$

Thus

$$\frac{1}{k_i} - FT_{Fi} \cos^3\theta - H \cos\phi\left(D_{iM} + T_{Hi} \cos^2(\phi - \omega D_{iM})\right)$$

$$+ 2\omega T_{Fi}^2 F^3 \sin\theta + 2\omega T_{Hi}^2 H^3 \sin\phi > 0$$

The minimum of the left hand side is at $\omega = \theta = \phi = 0$. Thus

$$K_i(FT_{Fi} + HT_{Hi} + HD_{iM}) < 1 \qquad \dots (3.2.19)$$

Since $F \leq 1$, $H \leq 1$ and $K_i(T_{Fi} + T_{Hi} + D_{iM}) < \sqrt{2} - 1$, this is always satisfied.

Repeating this with the filter G(s) shows that expr. (*3.2.17*) is a sufficient condition for the double-ended system, subject to the restriction of first-order lowpass filters in the control loops, as in the derivation of inequality (*3.2.13*).

3.3 Stability of sampled single-ended systems

For sampled systems also, the stability may be deduced by means of the diagonal dominance method. It is necessary to show that the matrix $\underline{A}^*(z)$ in eqn. (*2.2.7*) is diagonally dominated for all $|z| \geq 1$. By the argument of section 3.1.2, this reduces to the condition that $|\theta(z)| < 1$ for such values of z, as in expr. (*3.1.2*). Since $|\theta(z)| \to 0$ as $|z| \to \infty$, it is clear that the z-plane outside the unit circle maps via $\theta(z)$ into the inside of $\theta(e^{j\phi})$. Thus there are no poles of $\theta(z)$ for $|z| > 1$ and by the maximum modulus theorem it is possible to restrict the examination of the diagonal dominance to $|z| = 1$.

For $z = 1$, $\theta(z) = 1$. This corresponds to the lack of an external reference frequency; the system frequency is a function of the system parameters.

For the single-ended system, the general equations (*2.2.6*) are modified by putting $H^*_{ij}(z) = 0$. The stability condition now becomes

$$\left| 1 + \frac{T_s}{z-1} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} . F^*_{ij}(z) \right| > \left| \frac{T_s}{z-1} \right| \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} . \left| F^*_{ij}(z) . z^{-D_{ij}} \right| \quad \ldots \quad (3.3.1)$$

for $i = 1, 2, \ldots, n$ and $|z| = 1$ where $z \neq 1$. The delay term can, therefore, be deleted. In the subsequent analysis it will be convenient to use

$$K_i = T_s \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}$$

The results in this section have already been published[35], but with abbreviated derivation. A reprint is appended to this Thesis.

3.3.1 No delays or filters

Here expr. (*3.3.1*) becomes

$$|z - 1 + K_i| > K_i \quad \text{for } |z| = 1,\ z \neq 1.$$

For such z, $|z - 1 + K_i| = K_i$ is a circle with its centre at the point $(1 - K_i, 0)$, which passes through the point (1, 0) when $z = 1$. This circular locus clearly lies within the unit circle if $K_i < 1$. When the inequality applies, the locus of $\theta(z)$ is also within the unit circle and stability is assured.

3.3.2 Effect of delays

Since the inequality (*3.3.1*) holds for all $|z| \geq 1$ if it is true for $|z| = 1$, it is obvious that the delay term in that inequality does not affect the stability. The sampled single-ended system without filters is thus stable if for each i

$$\sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} < \frac{1}{T_s} \qquad \ldots \quad (3.3.2)$$

whatever the line delays. As $T_s \rightarrow 0$ and the system approaches the continuous system described in section 3.1.2, the upper bound on the gains tends to infinity. In all cases it will be shown that the conditions approach the continuous system conditions as the sampling period becomes smaller.

3.3.3 Effect of filters

Here is is assumed that the filters are of analogue type with transfer functions $F_{ij}(s) = F_i(s) = 1/(1 + sT_{Fi})$. The Z-transform of the filter response is

$$F^*_i(z) = \frac{w}{T_{Fi}} \left(\frac{1}{z - e^{-1/y_i}}\right) \quad \text{where } y_i = \frac{T_{Fi}}{T_s}$$

It will be convenient to put $f = e^{-1/y_i}$ and $F = \frac{w}{y_i} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}$.

The constant w is the fraction of the period T_s for which the sampling switches are closed, and is known as the sampler gain. The diagonal dominance condition is now

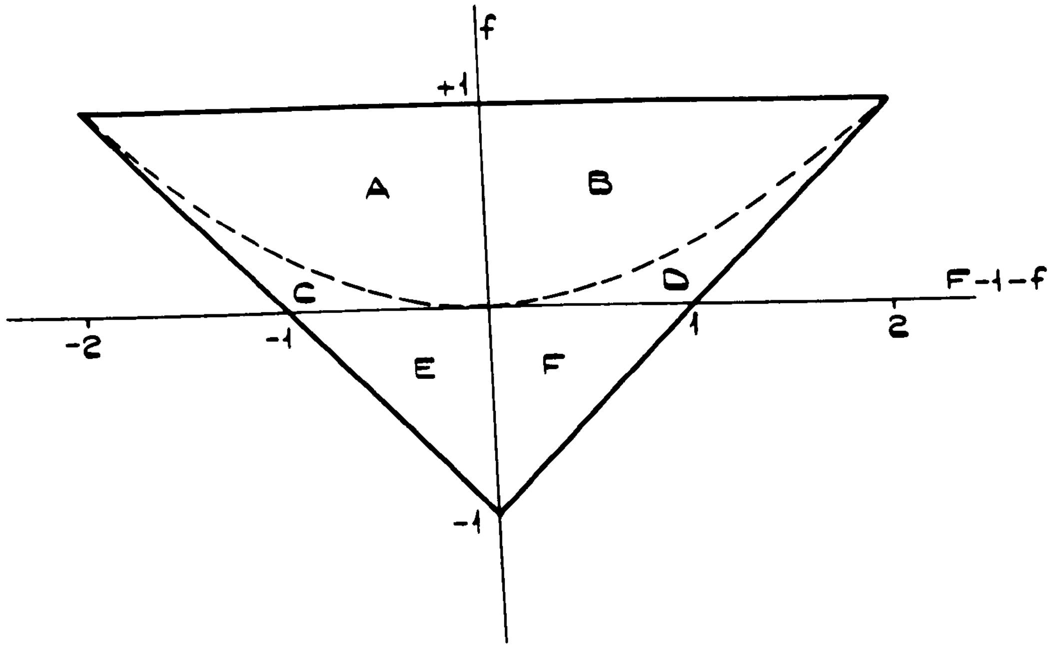
$$|(z - 1)(z - f) + Fz| > F \qquad \ldots \quad (3.3.3)$$

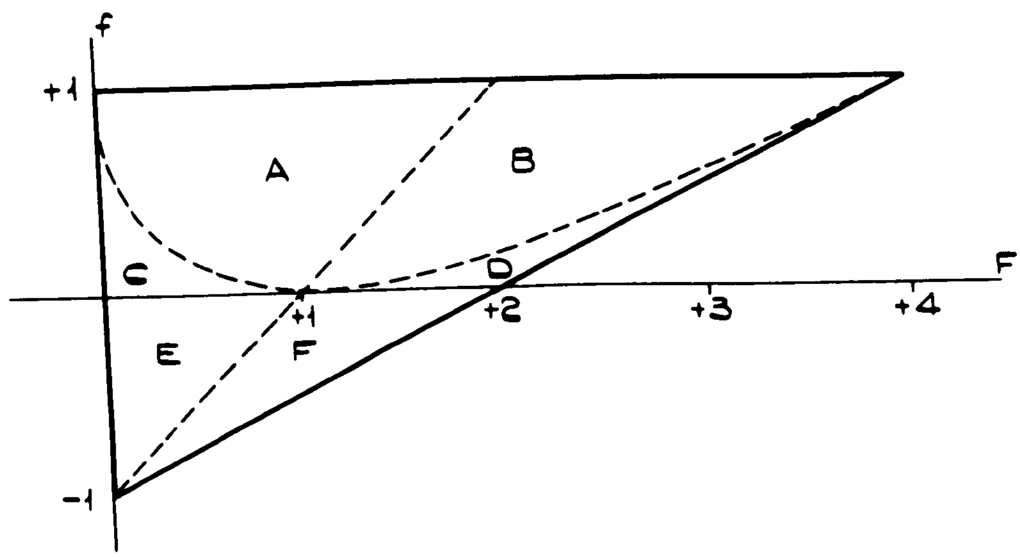
for $|z| = 1,\ z \neq 1$.

The quadratic function of z, $(z^2 - (1 + f - F)z + f)$ must have no roots for $|z| > 1$. The numbers F and f are both positive; f is less than unity. For the roots of the quadratic to lie in $|z| < 1$, the coefficients are such that F must be less than $2(1 + f)$ (see fig. 3.7). With this condition satisfied, the maximum modulus theorem holds. The modulus of the quadratic function is the product of the distances from the roots to the value of z in question. For $|z| = 1$, this is illustrated in fig. 3.8.

The roots, if real, must both have the same sign since f is positive. If both are real and negative, the minimum of the left-hand side of expr. (3.3.3) will occur at $z = -1$, and $z = +1$ will be a maximum. Since at $z = +1$ the expression becomes an identity, for

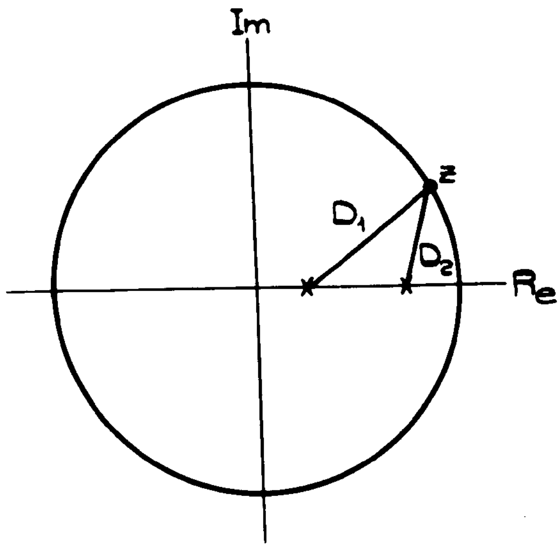
$z^2+(F-1-f)z+f=0$ has roots in $|z|<1$ if coefficients lie within indicated areas.



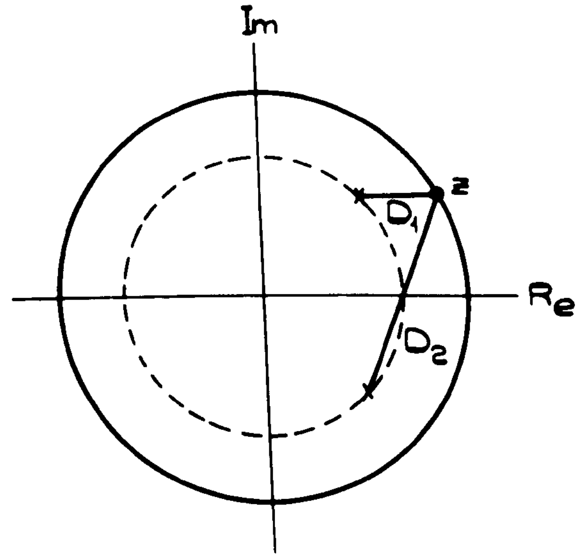


Area :- A Complex roots, negative real parts
B Complex roots, positive real parts
C Real roots, both negative
D Real roots, both positive
E, F } Real roots, one negative, one positive

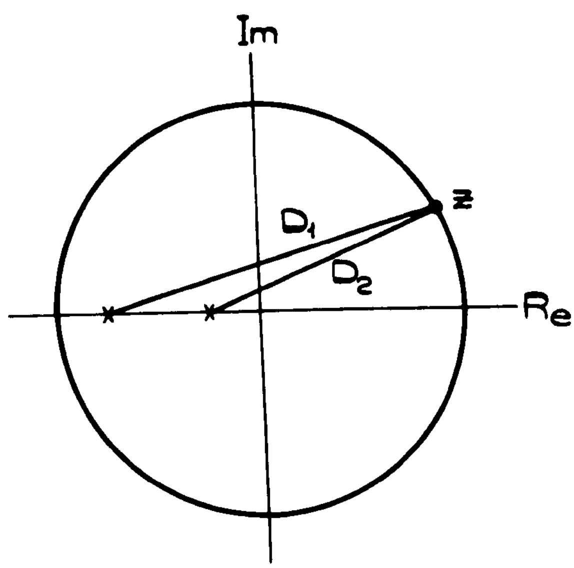
FIG. 3.7 SAMPLED SINGLE-ENDED SYSTEM: LIMITS OF COEFFICIENTS.



(a) Real Positive Roots



(b) Complex Roots



(c) Real Negative Roots

Note : The product of the lengths $D_1 . D_2$ must be minimised.

FIG. 3.8. SAMPLED SINGLE-ENDED SYSTEM : POSSIBLE ROOTS OF CHARACTERISTIC EQUATION.

all other z on the unit circle it would not be satisfied. If both roots are real and positive, $z = +1$ is clearly the minimum of the left-hand side of expr. (3.3.3). Thus the condition for these roots would be one sufficient conditon for stability. This condition is $F < (1 - \sqrt{f})$. However, the situation for complex roots gives a larger limit to F.

When $F < (1 + f)$, the complex roots lie in the right-hand half-plane on the circle of radius $\sqrt{f}$, at a distance $\sqrt{F}$ from the point $z = 1$, as shown in fig. 3.9. Where the roots are $\sqrt{f}\, e^{\pm j\phi}$ and $z = e^{j\theta}$, the square of the modulus of the left-hand side of expr. (3.3.3) may be evaluated by the cosine formula. Referring to fig. 3.9 this is

$$M^2 = x_1{}^2 \cdot x_2{}^2 = (1 + f)^2 - 2\sqrt{f}(1 + f) \cdot 2\cos\phi.\cos\theta + 2f(\cos 2\phi + \cos 2\theta) \qquad \ldots \quad (3.3.4)$$

Differentiating this with respect to θ

$$\frac{d(M^2)}{d\theta} = 4\sqrt{f}(1 + f)\cos\phi.\sin\theta - 4f\sin\theta.\cos\phi \qquad \ldots \quad (3.3.5)$$

$$= 0 \text{ when } \sin\theta = 0 \text{ or } \cos\theta = \frac{1 + f}{2\sqrt{f}}\cos\phi$$

$$\text{Now } \cos\phi = \frac{1 + f - F}{2\sqrt{f}} \text{ so } \cos\theta = \frac{(1 + f)(1 + f - F)}{4f}$$

For complex roots in the right-hand half-plane, it is clear that the modulus is a maximum at $z = -1$. Thus if the turning point given by this value of cosθ exists, then the modulus is a maximum at $z = +1$ also. For stability, however, it is necessary that $z = +1$ give a minimum. Thus the system is stable if

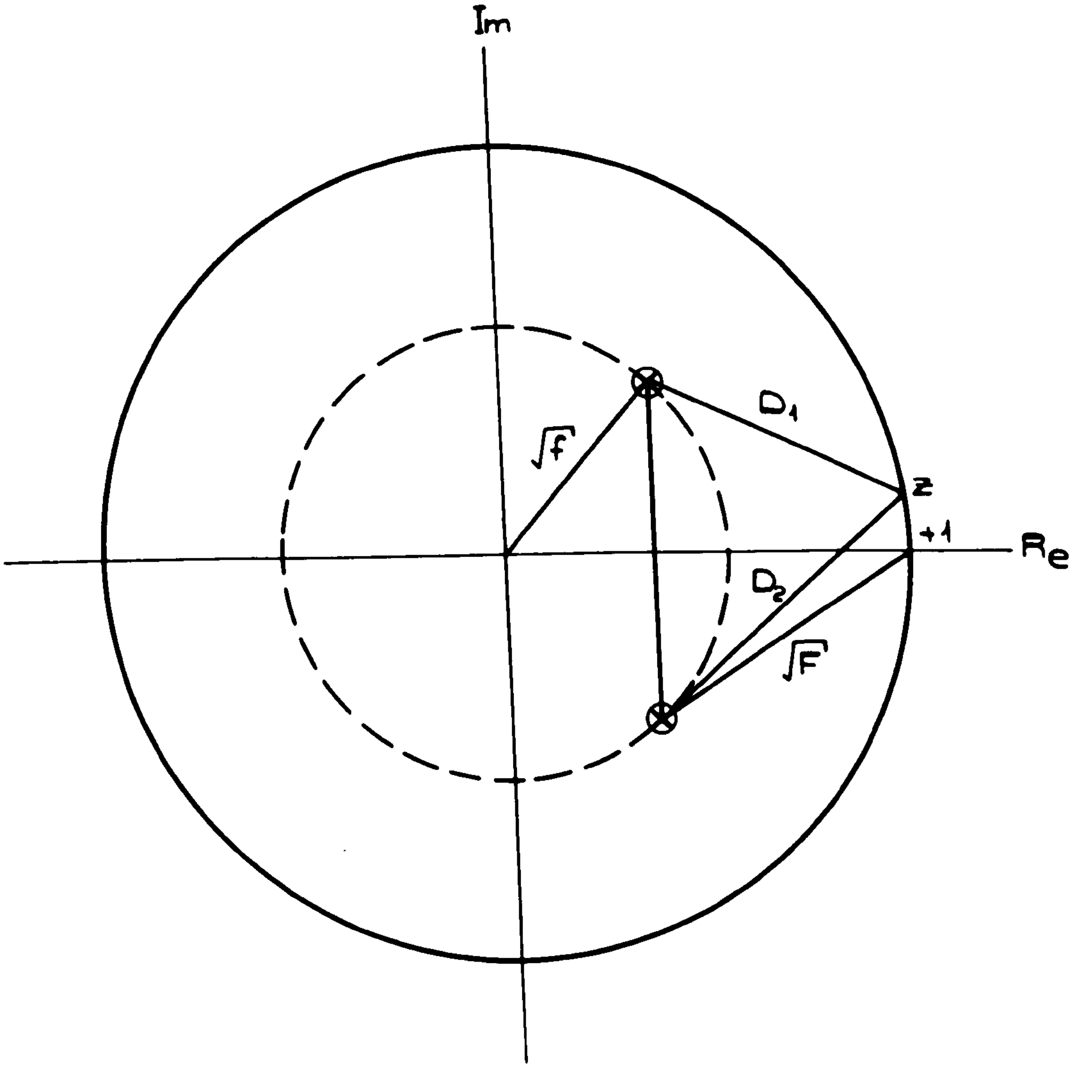


FIG. 3.9 STABILITY DIAGRAM : SAMPLE SINGLE-ENDED SYSTEM WITH FILTER.

$$\frac{(1 + f)(1 + f - F)}{4f} > 1 \text{ which reduces to}$$

$$F < \frac{(1 - f)^2}{1 + f} \qquad \ldots \quad (3.3.6)$$

This is the condition given in the published paper [35].

It is interesting to investigate the behaviour of this function as $T_s \to 0$. The condition is

$$\frac{w}{T_{Fi}} T_s \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} < \frac{(1 - e^{-T_s/T_{Fi}})^2}{1 + e^{-T_s/T_{Fi}}} \to \tfrac{1}{2}\left(\frac{T_s}{T_{Fi}}\right)^2$$

Thus, in the limit

$$\left(\frac{w}{T_s}\right) T_{Fi} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} = T_{Fi} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} < \tfrac{1}{2}$$

which is the expression for the single-ended continuous system, *(3.1.7)*. The condition for the roots to be real corresponds to the condition for no transient overshoot; the right-hand side of the inequality is reduced to $\frac{1}{2}$.

3.4 Stability of sampled double-ended systems

This, as may be expected from the previous sections, is by far the most complicated case. The first case to be examined is that of a system with no filters. This is followed by a special case in which $F^*_{ij}(z) = H^*_{ij}(z) = F^*_i(z)$. The result for this case has proved easier to find than that for distinct filters $F^*_i(z)$ and $H^*_i(z)$.

The maximum modulus theorem holds for each result, as indicated earlier, since the results are deduced by the diagonal dominance method. All the results are shown to reduce to the continuous system criteria as $T_s \to 0$.

The stability condition is that for $|z| = 1$

$$\left| z - 1 + T_s \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} F^*_{ij}(z) + H^*_{ij}(z).z^{-(\delta_{ij}+D_{ji})} \right|$$

$$> T_s \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \left\{ |F^*_{ij}(z)| + |H^*_{ij}(z)| \right\}$$

$$> T_s \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \left| F^*_{ij}(z).z^{-D_{ij}} + H^*_{ij}(z).z^{-\delta_{ji}} \right| \qquad \ldots \quad (3.4.1)$$

The method used to deduce the following results follows investigation of the two sides of the diagonal dominance condition separately. If, at the point $z = +1$, the radius of curvature of the left-hand function is greater than that of the right-hand function, then there is a local maximum of the function $\theta(z)$ in the region of that point. The right-hand side will be seen to have a single maximum at $z = +1$ and a single minimum at $z = -1$, in the only case where it varies at all, for distinct $F_i^*(z)$ and $H^*_i(z)$. It has to be shown that the left-hand function has a minimum at $z = +1$ for the condition to be correct.

3.4.1 Effect of delays alone

Putting $F^*(z) = H^*(z) = 1$, $K = T_s \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}$, and

$D_i = (\delta_{ij} + D_{ji})_{\max\, j}$, it is clear that, for $z = e^{j\theta}$, the vector

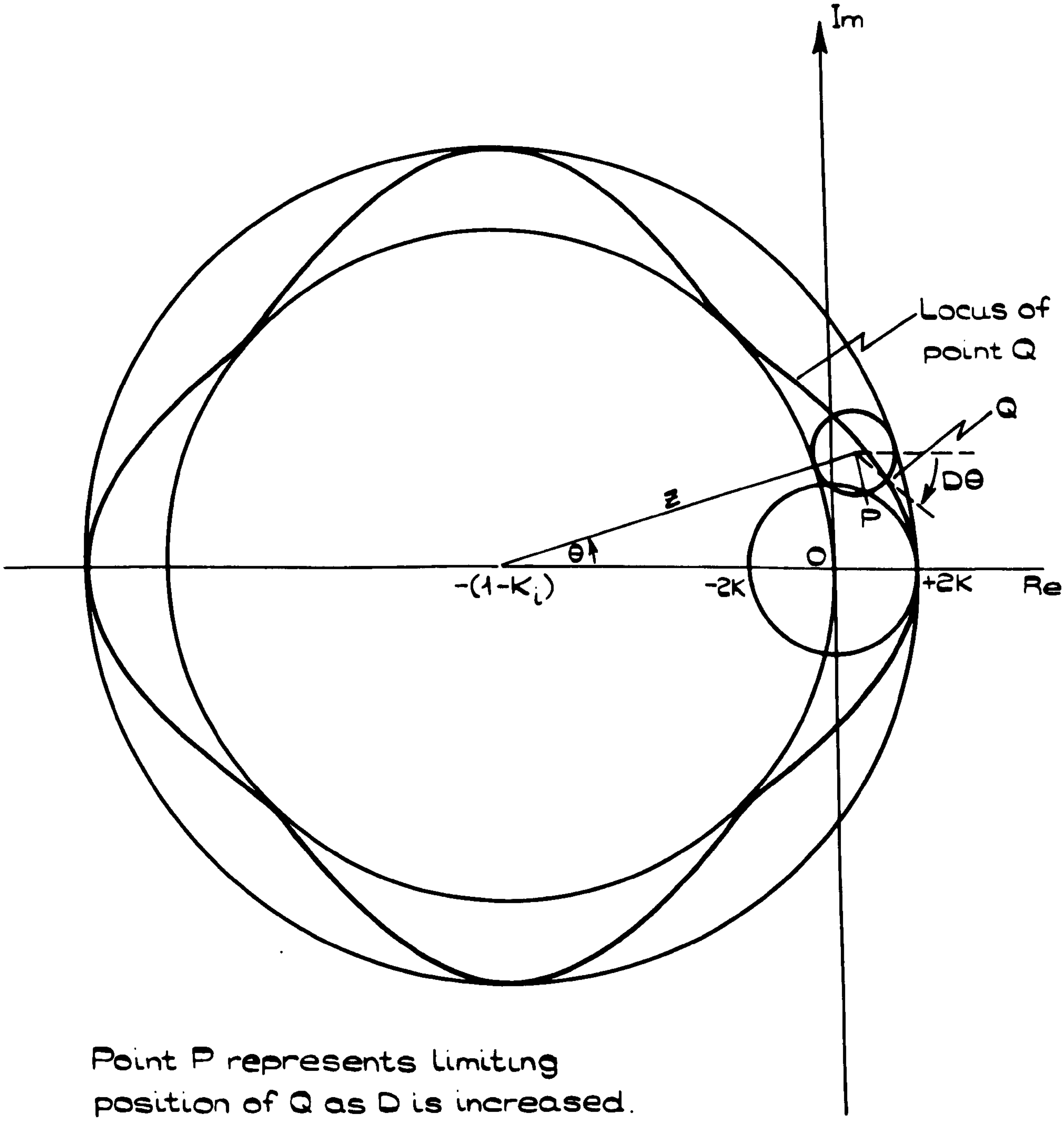
$T_s \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}.z^{-(\delta_{ij} + D_{ji})}$ lies within the sector of the circle with centre at at the origin and radius K_i which is bounded by the vectors K_i and $K_i.z^{-D_i}$, when $D_i\theta < \frac{1}{2}\pi$.

Thus expr. (3.4.1) may be rewritten as

$$|z - 1 + K_i + K_i.z^{-D_i}| > 2K_i \qquad \ldots \quad (3.4.2)$$

Examining $\psi(z) = -(1 - K_i) + z + K_i.z^{-D_i}$, in fig. 3.10, it will be seen that this describes a path lying between the concentric circles with centres at $(-1 + K_i, 0)$ and radii $1 + K_i$ and $1 - K_i$. The path touches the outer circle when $(D_i + 1)\theta = 2n\pi$ and the inner circle when $(D_i + 1)\theta = (2n + 1)\pi$.

For $0 \leq (D_i + 1)\theta \leq \pi$, the locus of $\psi(z)$ has its greatest curvature at $\theta = 0$. If at this point, the radius of curvature is



Point P represents limiting position of Q as D is increased.

FIG. 3.10. STABILITY DIAGRAM : SAMPLED DOUBLE-ENDED SYSTEM WITHOUT FILTERS.

greater than $2K_i$, then the locus lies outside the circle with centre at $(-K_i, 0)$ and radius $2K_i$. Thus expr. *(3.4.2)* is satisfied and so is expr. *(3.4.1)*.

When $z = e^{j\theta}$ and $\theta \to 0$, $\psi(z)$ may be written

$$\psi(z) = A - \tfrac{1}{2}B\theta^2 + jC\theta$$

where $A = 2K_i$, $B = 1 + K_iD_i^2$, $C = 1 - K_iD_i$. For small θ, the radius of the curvature of $\psi(z)$ is

$$R = \frac{C^2}{B} + \tfrac{1}{4}B\theta^2$$

The condition is satisfied if $R > 2K_i$, i.e. if $C^2 > 2B.K_i$ for which

$$K_iD_i < \sqrt{2 + \frac{2}{D_i} + (\frac{1}{D_i})^2} - (1 + \frac{1}{D_i}) \qquad \ldots \quad (3.4.3)$$

The right-hand side is less than $\sqrt{2} - 1$ for non-zero D_i.

As $T_s \to 0$, $1/D_i \to 0$ but $T_sD_i \to D_{iM}$ as defined by section 3.2.2 and this expr. *(3.4.3)* becomes expr. *(3.2.7)*, the stability condition for the continuous system.

3.4.2 Effect of a single filter

The condition for the stability of a sampled system is given by expr. *(3.4.1)*. This is rather complex so this section investigates the simpler case of $F^*_{ij}(z) = H^*_{ij}(z) = F^*_i(z)$ for all i, j. The form of $F^*_i(z)$ is identical to that used in section 3.3.3.

Subject to the conditions for the use of the maximum delay, D_i, the condition for the system stability by the diagonal dominance method becomes

$$\left|(z - 1) + \left(\frac{F_i z}{z - f}\right)\left(1 + z^{-D_i}\right)\right| > 2F_i\left|\frac{z}{z - f}\right|$$

which may be expressed as

$$|(z - 1)(z - f) + F_i z(1 + z^{-D_i})| > 2F_i \qquad \ldots \quad (3.4.4)$$

for $|z| = 1$. This may also be treated as $|\psi(z)| > 2F_i$ (see fig. *3.11*).

For small θ where $z = e^{j\theta}$, $\psi(z) = A - \frac{1}{2}\theta^2 B + j\theta C$, where $A = 2F_i$, $B = 3 - f + F_i(2 - 2D_i + D_i^2)$ and $C = 1 - f + F_i(2 - D_i)$.

Again, the radius of curvature of the locus of $\psi(z)$ must be greater than $2F_i$ at $z = +1$. The expression $C^2 > B.2F_i$ reduces to

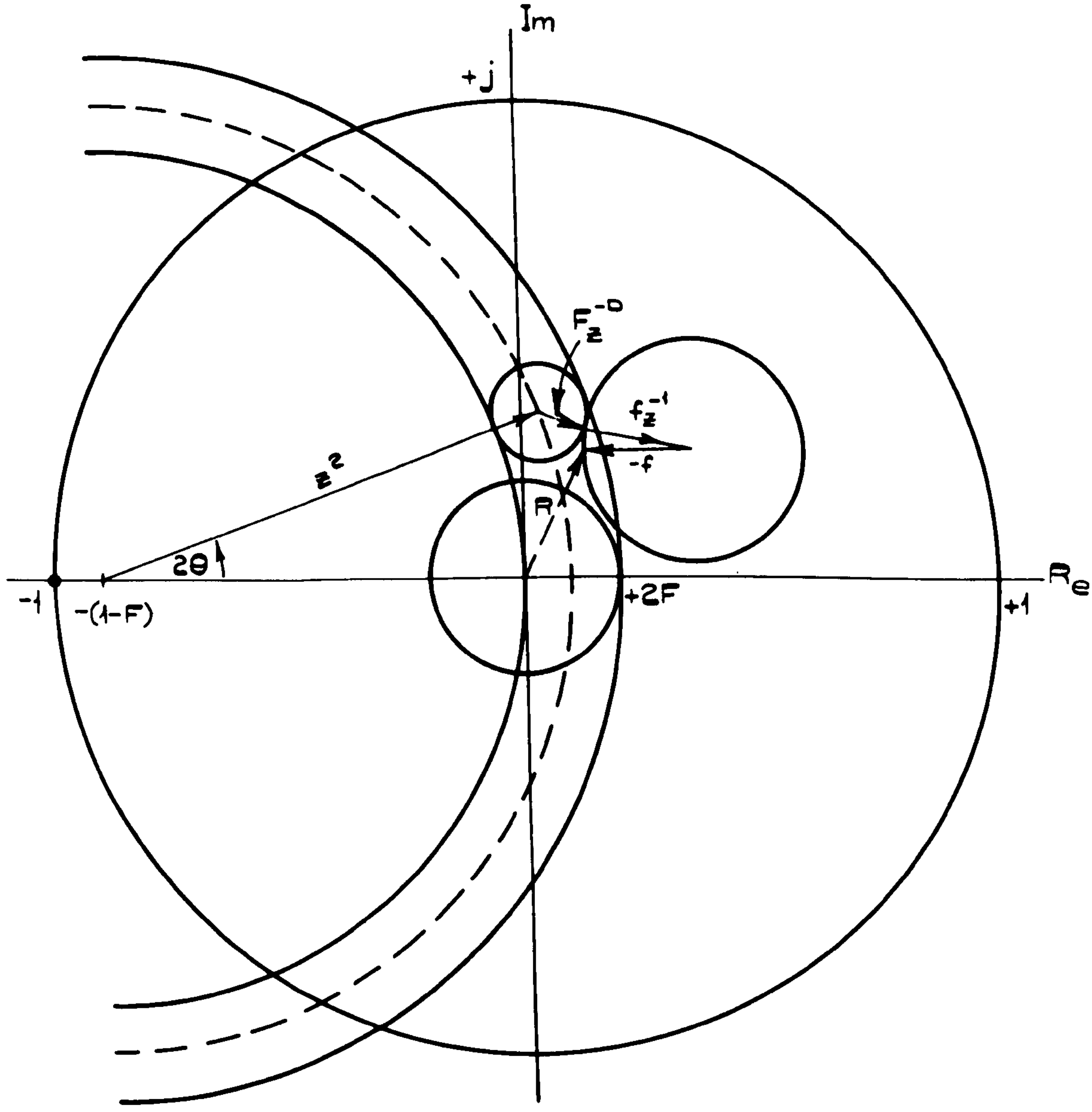
$$\left\{\frac{F_i.D_i}{1 - f}\right\}^2 + 2\left\{\frac{F_i.D_i}{1 - f}\right\}\left\{1 + \frac{1 + f}{D\ (1 - f)}\right\} - 1 < 0 \qquad \ldots \quad (3.4.5)$$

This gives an upper bound on F_i, namely

$$\frac{F_i\ D_i}{1 - f} < \sqrt{2 + 2\frac{(1 + f)}{D_i(1 - f)} + \left\{\frac{1 + f}{D_i(1 - f)}\right\}^2} - \left\{1 + \frac{1 + f}{D_i(1 - f)}\right\} \qquad \ldots \quad (3.4.6)$$

This result may be seen to include expr. (*3.4.3*); when $D_i = 0$, it is related to expr. (*3.3.6*) where $2F_i = K_i$.

For larger θ, it is necessary to differentiate $|\psi(z)|$ with respect to θ, to show that $\psi(z)$ always remains outside the circle of radius $2F_i$ which has its centre at the origin. This is similar to the result in section 3.2.5. For $D_i\theta < \frac{1}{2}\pi$ all the terms change so



$$R = \left| -(1-F) + z^2 + F_z^{-D} + f_z^{-1} - f \right| > 2F \quad \text{for stability}$$

FIG. 3.11 STABILITY DIAGRAM: SAMPLED DOUBLE-ENDED SYTEM WITH ONE FILTER.

as to increase the modulus $|\psi(z)|$ from its initial value at $\theta = 0$. For $D_i\theta > \frac{1}{2}\pi$ the geometry is such that if $|\psi(z)| \leq 2F_i$ the stability condition (3.4.6) is violated in any case. Thus the system is stable if expr. (3.4.6) is satisfied at every exchange.

By letting $T_s \to 0$ in expr. (3.4.6) it is possible to check the condition against that for the continuous system. It will be seen that the resulting condition is better than expr. (3.2.12) with $T_G = 0$ and $T_{Hi} \to T_{Fi}$. It can be seen that it includes this expression.

Now $F_i = \left(\frac{w}{T_s}\right)\left(\frac{T_s}{T_{Fi}}\right) T_s \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}$ and $D_i = \left[1 + \frac{D_{iM}}{T_s}\right]$ where D_{iM} is as defined in section 3.2.2 and $[\ ]$ is the Gaussian bracket notation (see eqn. (2.2.6)).

Letting $T_s \to 0$ with $\frac{w}{T_s} \to 1$, $F_iD_i \to \frac{T_s}{T_{Fi}} D_{iM} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}$. However, $(1 - f) \to \frac{T_s}{T_{Fi}}$ so the left-hand side of expr. (3.4.6) becomes $D_{iM} \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}$. Now $(1 + f) \to 2$ as $T_s \to 0$ so

$$\frac{1 + f}{D_i(1 - f)} \to \frac{2}{\left[1 + D_{iM}/T_s\right]\left\{T_s/T_{Fi} + \frac{1}{2}(T_s/T_{Fi})^2 + \ldots\right\}} \to \frac{2T_{Fi}}{D_{iM}}$$

Thus in the limit as $T_s \to 0$ expr. (3.4.6) becomes

$$(D_{iM} + 2T_{Fi}) \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} < \left\{\frac{D_{iM} + 2T_{Fi}}{D_{iM}}\right\} \left\{\sqrt{1 + \left(\frac{D_{iM} + 2T_{Fi}}{D_{iM}}\right)^2} - \left(\frac{D_{iM} + 2T_{Fi}}{D_{iM}}\right)\right\} \quad \ldots \quad (3.4.7)$$

It is useful to consider $\phi(x) = x\{\sqrt{1 + x^2} - x\}$ where $x = (D_{iM} + 2T_{Fi})/D_{iM}$. As $x \to \infty$, $\phi(x) \to \frac{1}{2}$. At $x = 1$, $\phi(x) = \sqrt{2} - 1$. For all $x > 0$, $\frac{d\phi(x)}{dx} > 0$. Now $x = 1$ represents the case of no filters, and expr. *(3.4.7)* is identical to expr. *(3.2.7)*. $x = \infty$ represents the case of no delays and expr. *(3.4.7)* becomes expr. *(3.2.11)*. For values of delay and filter time-constant between these extremes expr. *(3.4.7)* is better than expr. *(3.2.12)* for this special case.

3.4.3 Delays and separate filters F* and H*

This is by far the most complicated case, simply because of the amount of manipulation required. The stability condition is that for all i, with $|z| = 1$,

$$\left|(z - 1) + \frac{F_i.z}{z - f} + \frac{F_i.z}{z - h} . z^{-D_i}\right| > \left|\frac{F_i.z}{z - f}\right| + \left|\frac{F_i.z}{z - h}\right|$$

which may be expressed as

$$|(z - 1)(z - f)(z - h) + F_i z(z - h) + H_i z^{1-D_i}(z - f)|$$

$$> F_i|z - h| + H_i|z - f| \qquad \ldots \quad (3.4.8)$$

The right-hand side has increasing value from $z = +1$ to $z = -1$, $|z| = 1$. Letting M_1 denote the left-hand side and M_2 the right-hand side, then

$$M_2^2 = F_i^2 + H_i^2 + (F_i^2h^2 + H_i^2f^2) - 2(F_i^2h + H_i^2f)\cos\theta$$

$$+ 2F_iH_i(1 - 2h\cos\theta + h^2)^{\frac{1}{2}}(1 - 2f\cos\theta + f^2)^{\frac{1}{2}}$$

But as $\theta \to 0$, from eqn. *(3.4.1)*,

$$\frac{d}{d\theta}(M_2^2) = 2\theta\{F_i^2h + H_i^2f + F_iH_i(f + h)\}$$

As $\theta \to 0$, $\frac{d}{d\theta}(M_2^2) \to \theta \frac{d^2}{d\theta^2}(M_2^2)$; this property is shared by the derivatives of M_1^2 also.

Now $$M_1 = |z^3 + (F_i - 1 - f - h)z^2 + (f + h + fh - F_i h)z - fh + H_i z^{1-D_i}(z - f)|$$

Thus $$M_1^2 = \{\cos 3\theta + (F_i - 1 - f - h)\cos 2\theta + (f + h + fh - F_i h)\cos\theta - fh + H_i \cos(D_i - 2)\theta - H_i f\cos(D_i - 1)\theta\}^2$$
$$+ \{\sin 3\theta + (F_i - 1 - f - h)\sin 2\theta + (f + h + fh - F_i h)\sin\theta - H_i \sin(D_i - 2)\theta + H_i f\sin(D_i - 1)\theta\}^2$$
$$= U^2(\theta) + V^2(\theta)$$

Differentiating this

$$\frac{d}{d\theta}(M_1^2) = -2U(\theta)\{3\sin 3\theta + 2(F_i - 1 - f - h)\sin 2\theta + (f + h + fh - F_i h)\sin\theta + H_i(D_i - 2)\sin(D_i - 2)\theta - H_i f(D_i - 1)\sin(D_i - 1)\theta\}$$
$$+2V(\theta)\{3\cos 3\theta + 2(F_i - 1 - f - h)\cos 2\theta + (f + h + fh - F_i h)\cos\theta - H_i(D_i - 2)\cos(D_i - 2)\theta + H_i f(D_i - 1)\cos(D_i - 1)\theta\}$$

In the expansion of this expression, it is immediately obvious that all the terms of the form $2.\sin r\theta.\cos r\theta$ in the first product cancel the similar terms in the second product. The remaining terms can be combined to form terms of the form $\sin(r - s)\theta$. The expression now becomes

$$\frac{d}{d\theta}(M_1^2) = 2fh\Big\{3\sin 3\theta + 2(F_i - 1 - f - h)\sin 2\theta + (f + h + fh - F_i h)\sin\theta$$
$$+ H_i(D_i - 2)\sin(D_i - 2)\theta - H_i f(D_i - 1)\sin(D_i - 1)\theta\Big\}$$
$$+ 2\Big\{- 3(F_i - 1 - f - h)\sin\theta - 3(f + h + fh - F_i h)\sin 2\theta$$
$$- 3H_i\sin(D_i + 1)\theta + 3H_i f\sin(D_i + 2)\theta$$
$$+ 2(F_i - 1 - f - h)\sin\theta$$
$$- 2(F_i - 1 - f - h)(f + h + fh - F_i h)\sin\theta$$
$$- 2H_i(F_i - 1 - f - h)\sin D_i\theta$$
$$+ 2H_i f(F_i - 1 - f - h)\sin(D_i + 1)\theta$$
$$+ (f + h + fh - F_i h)\sin 2\theta$$
$$+ (F_i - 1 - f - h)(f + h + fh - F_i h)\sin\theta$$
$$- H_i(f + h + fh - F_i h)\sin(D_i - 1)\theta$$
$$+ H_i f(f + h + fh - F_i h)\sin D_i\theta$$
$$- H_i(D_i - 2)\sin(D_i + 1)\theta$$
$$- H_i(D_i - 2)(F_i - 1 - f - h)\sin D_i\theta$$
$$- H_i(D_i - 2)(f + h + fh - F_i h)\sin(D_i - 1)\theta$$
$$- H_i^2 f(D_i - 2)\sin\theta + H_i f(D_i - 1)\sin(D_i + 2)\theta$$
$$+ H_i f(D_i - 1)(F_i - 1 - f - h)\sin(D_i + 1)\theta$$
$$+ H_i f(D_i - 1)(f + h + fh - F_i h)\sin D_i\theta$$
$$+ H_i^2 f(D_i - 1)\sin\theta\Big\}$$

Now for stability it is necessary that in the region of $z = +1$

$$\frac{d}{d\theta}(M_1^2) > \frac{d}{d\theta}(M_2^2)$$

This reduces to

$$3fh\sin 3\theta + 2\{fh(F_i - 1 - f - h) - (f + h + fh - F_i h)\}\sin 2\theta$$

$$+ \{(1 + f + h + fh - F_i)(f + h + fh - F_i h) + H_i^2 f - (F_i - 1 - f - h)\}\sin\theta$$

$$+ H_i fh(D_i - 2)\sin(D_i - 2)\theta$$

$$- \{H_i f^2 h + H_i(f + h + fh - F_i h)\}(D_i - 1)\sin(D_i - 1)\theta$$

$$+ H_i D_i\{f(f + h + fh - F_i h) - (F_i - 1 - f - h)\}\sin D_i\theta$$

$$+ H_i(D_i + 1)\{f(F_i - 1 - f - h) - 1\}\sin(D_i + 1)\theta$$

$$+ H_i f(D_i + 2)\sin(D_i + 2)\theta$$

$$> \{F_i^2 h + H_i^2 f + F_i H_i(f + h)\}\sin\theta \qquad \ldots \; (3.4.9)$$

Thus as $\theta \to 0$

$$9fh + 4\{fh(F_i - 2 - f - h) - (f + h + F_i h)\}$$

$$+ \{(1 + f + h + fh)(f + h + fh - F_i h) + (1 + f + h - F_i)$$

$$- F_i(f + h + fh)\}$$

$$+ H_i fh(D_i - 2)^2 - (D_i - 1)^2 H_i(f + h + fh + f^2 h - F_i h)$$

$$+ H_i D_i^2\{f(1 + f + h + fh - F_i h) + 1 + h - F_i\}$$

$$+ H_i(D_i + 1)^2\{f(F_i - 1 - f - h) - 1\} + H_i f(D_i + 2)^2$$

$$> F_i H_i(f + h) \qquad \ldots \; (3.4.10)$$

This reduces to

$$(1 - f)^2(1 - h)^2 - F_i(1 + f)(1 - h)^2 - H_i(1 + h)(1 - f)^2$$

$$- F_i H_i D_i^2(1 - f)(1 - h) - 2H_i D_i(1 - f)^2(1 - h) + 2F_i H_i D_i(f - h)$$

$$> 0 \qquad \ldots \; (3.4.11)$$

When this is written in the form

$$\frac{F_i H_i D_i^2}{(1 - f)(1 - h)} + \frac{F_i(1 + f)}{(1 - f)^2} + \frac{H_i D_i}{(1 - h)}\left\{2 + \frac{1 + h}{(1 - h)D_i}\right\}$$

$$+ \frac{2(h - f)F_i H_i D_i}{(1 - f)^2(1 - h)^2} < 1 \qquad \ldots \textit{(3.4.12)}$$

it can be seen to be of the same form as expr. (*3.4.5*). Relations such as expr. (*3.2.6*) may be derived by letting $T_s \rightarrow 0$.

This result is as far as the present investigation has gone. It remains to be proved that in fact there are no points at which the two moduli M_1 and M_2 are equal, other than at $z = +1$. If it can be shown rigorously that expr. (*3.4.12*) is a sufficient condition for the stability of a sampled double-ended system, most of the other criteria derived in this Chapter may be obtained from it.

3.5 Conclusion

It has now been shown that stability criteria are available for a large number of control schemes. The advantages of these criteria over other more exact ones is that each exchange may be considered separately and detailed knowledge of the network configuration is not required. It is also possible to use automatic means to maximise the gain used on each input whatever the number of working inputs at an exchange. This will cut down the time taken to absorb transient disturbances.

It would be useful to obtain the necessary and sufficient condition for the stability of a double-ended fully-interconnected system with filters. Some practical work by J. R. Jarvis has suggested that the presence of the filters gives a larger gain margin than the condition derived in section 3.2.3 would suggest. Unfortunately there is some doubt about the accuracy of the experimental results, which have not been repeated. A field-trial is, however, proposed and could provide a check of this work.

In the double-ended system a data link is needed between exchanges. This will be provided over the p.c.m. highway and will thus have a limited bandwidth. The signals can be either coarsely quantised or subject to an increased sampling period. In the former event the filters in the receiving terminals will need longer time-constants. In either case, stability requirements will cause the control system gain to be reduced, thus restricting the spread of initial frequencies that can be controlled. The requirements of the data channel are thus highly dependent on the quality of the oscillators employed.

4 SYSTEM OPERATING FREQUENCY

The purpose of the control system is to produce a stable operating frequency. The basic oscillators will be of a moderate stability design, and the injection of the control signals will be arranged in such a way that this basic stability is unimpaired. If the factors governing the controlled final frequency are known, the controller can be designed so that the frequency stability is maximised.

Provided that the system stability is proven, the Laplace Final Value theorem may be applied to the transformed equations for the continuous systems to obtain the final value of each frequency component. By differentiation of equations (*2.1.4*) for the single-ended system or (*2.1.8*) for the double-ended system, it will be seen that when these frequency components are equal the rate of change of each component is zero. Thus the frequencies asymptotically approach a common value, and it is only necessary to derive the expression for one component of the frequency vector.

The method for evaluating this expression is common to both the single- and double-ended systems. The results for these will be given separately below.

The matrix functions of s may be decomposed into a number of other matrices which are independent of s. For example

$$\underline{B}(s) = \underline{B}_0 + s\underline{B}_1 + s\underline{B}_2 + \dots$$

where $\underline{B}_0 = \underline{B}(0)$. Similarly, the filter transfer functions may be expressed as power series, e.g.

$$F_{ij}(s) = F_{ij}(0) - sT_{Fij} + \text{higher powers of } s$$

When $F_{ij}(s)$ represents a first order lowpass filter, T_{Fij} is the time constant of the filter. For higher order filters the following results still apply with the equivalent constants defined above.

Cramer's rule will be applied to obtain the expression for the frequency components. In the case of the double-ended system defined by the equations *(2.1.9)*, the matrix $\underline{\beta}_\xi(s)$ is defined by the substitution of the column vector of the right hand side of the equations for the ξth column of the matrix $\{s\underline{I} - \underline{B}(s)\}$ in the system equations. This is decomposed as above for each $\xi = 1, 2, \ldots n$, and the final value of the ξth frequency element given by

$$\lim_{t\to\infty} \{f_\xi(t)\} = \lim_{s\to 0}\left\{s.\frac{\det.|\underline{\beta}_\xi(s)|}{\det.|s\underline{I} - \underline{B}(s)|}\right\}$$

$$= \frac{\det.|\underline{\beta}_\xi(0)|}{\lim\limits_{s\to 0}\left\{\frac{1}{s}.\det.|s\underline{I} - \underline{B}(s)|\right\}}$$

$$= \frac{\sum\limits_{i=1}^{n} B_{0i\xi}\{f_{0i} + v_i(0)\}}{\sum\limits_{i=1}^{n} B_{0ii} - \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} b_{1ij}B_{0ij}} \qquad \ldots \qquad (4.1)$$

where the cofactor of the element b_{mij} in $\underline{B}_m$ is denoted by B_{mij}.

In the Appendix (chapter 9) it will be seen that the cofactors $B_{0ij} = B_{0ik}$ for all j, k. The above expression is thus independent of ξ and equal to the final frequency f_s.

The Appendix also shows that for the special case where each input to the ith exchange has a common gain k_i, the cofactors are linked and can be eliminated from expression (4.1). The gains are

$$k_{ij} = c_{ij}.k_i$$

where the connection factors c_{ij} and c_{ji} are equal, taking value 1 when the ith and jth exchanges are directly connected, and zero otherwise. It is necessary that, for all i, j, $F_{ij}(0) = F$; G and H are similarly defined. With these restrictions it is shown that $k_i.B_{0ij}$ is a constant for all i, j. Thus the final frequency is given by

$$f_s = \frac{\sum_{i=1}^{n} \frac{1}{k_i}\left\{f_{oi} + v_i(0)\right\}}{\sum_{i=1}^{n} \frac{1}{k_i}\left\{1 - \sum_{j=1}^{n} b_{1ij}\right\}} \qquad \ldots \quad (4.2)$$

This equation is an extension of the results published by Brilliant[14], West[20] and Pierce[31] for the case where each exchange has a fixed gain for all inputs. The theorem given in the Appendix extends that given by Brilliant for the same restrictions.

4.1. Double-ended systems

On substituting the vector and matrix elements into equation (4.2) the expression for the final frequency becomes

$$f_s = \frac{\sum_{i=1}^{n} (f_{oi}/k_i) - (F-H)\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} + (F-H)(G-1)\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}.\phi_{oi}}{\sum_{i=1}^{n} (1/k_i) + \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{(G-1)(T_{Fij} - T_{Hij} - H.\tau_{ij}) + (F-H)(T_{Gij} + G.d_{ij})\right\}} \qquad \ldots \quad (4.1.1)$$

In this expression, $c_{ii} = 0$. The recovery circuits are such that $G = 1$, so

$$f_s = \frac{\sum_{i=1}^{n} (f_{oi}/k_i) - (F - H) \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}}{\sum_{i=1}^{n} (1/k_i) + (F - H) \sum_{i=1}^{n} \sum_{j=1}^{n} (T_{Gij} + d_{ij})} \qquad \ldots \quad (4.1.2)$$

is the final frequency of an unbalanced double-ended system.

The constants r_{ij} are affected by the operation mode of the system and the line delays d_{ij} will vary with temperature. It is thus likely that to maintain a stable operating frequency the controller will be of the double-ended type, balanced so that $F = H = 1$. The balanced system operating frequency is

$$f_s = \frac{\sum_{i=1}^{n} (f_{oi}/k_i)}{\sum_{i=1}^{n} (1/k_i)} \qquad \ldots \quad (4.1.3)$$

which is independent of delays or operating mode.

It will be noted that the only initial condition that affects the final frequency is the constant r_{ij}. The initial phases and data link delays are eliminated from the expressions in all cases. This point has not been noted explicitly by those others who have obtained results based on equation (4.2)[20, 31].

4.2 Single-ended systems

The single-ended system may be treated in exactly the same way as the double-ended system to obtain equation (4.2), modified by the substitution of $u_i(0)$ for $v_i(0)$ and a_{1ij} for b_{1ij}. It will be seen, however, that by putting $H = 0$, $v_i(0)$ reduces to $u_i(0)$ and b_{1ij} to a_{1ij}. Thus, with this substitution, equation (4.1.2) is also valid for the single-ended system.

Without loss of generality $F = 1$ so that the final frequency of the single-ended system is given by

$$f_s = \frac{\sum_{i=1}^{n} (f_{oi}/k_i) - \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}}{\sum_{i=1}^{n} (1/k_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} (T_{Gij} + d_{ij})} \qquad \ldots \quad (4.2.1)$$

In this case the delays and operating mode affect the frequency. Generally the network will be arranged so that all the delays are integer multiples of one frame period at this frequency (see Chapter 7). The integers $(-r_{ij})$ will then correspond, in the correct operating mode, to these integers. In the next Chapter, any variation of the delays from these ideal settings will be denoted by δ_{ij} and the $(n - 1)$ dimensional vector $\underline{\lambda}$ will be defined in terms of r_{ij}. This vector defines the operating mode and is zero in the correct mode. The sum of its elements λ_ν is only zero when all the δ_{ij} are zero. Substituting these new terms, the final frequency can be expressed as

$$f_s = \frac{\sum_{i=1}^{n} (f_{oi}/k_i) - \sum_{\nu=1}^{n-1} \lambda_\nu}{\sum_{i=1}^{n} (1/k_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} (\delta_{ij} + T_{Gij})} \qquad \ldots \quad (4.2.2)$$

Thus it is the variation of the delays from their nominal values that affects the operating frequency.

As the delays are varied, the frequency will be a piece-wise linear function of the delays. Under certain circumstances it is possible for the frequency to lie outside the controllable range of the oscillators. Use of the double-ended system minimises the risk of this happening once the control has been established.

4.3 Sampled systems

It may readily be seen, by application of the final value theorem for the Z-transform and the cofactors theorem given in the Appendix, that the final frequency of a sampled system without delays or filters is given by equation (*4.1.3*).

The line delays are not equal to integer multiples of the sampling period. This requires a correction term to be added to the vector $\underline{u}^*(z)$ in equation (*2.2.8*). This and the filter terms will probably cause modification of the expressions for the system final frequency.

The filters used in the stability analysis are of the analogue type. It is more likely that digital filters will be used in a practical system, and these will generate a new set of system equations. Once the proposed type of filter is chosen it will be necessary to evaluate the expression for the final frequency in order to minimise the sampling period. This is not within the scope of the work described in this Thesis.

5 STEADY STATE CONDITIONS AND WRONG MODE OPERATION

In the steady state, all the oscillator frequencies will attain a common value. It has already been shown that this is a function of the uncontrolled frequencies, the gains, the delays and the constants r_{ij}. The latter are non-linear functions of the initial phase differences.

Some residual error signal is required at each exchange to drive the oscillator frequency from its uncontrolled value to the steady state value. In the single-ended system, the error signals are the buffer-store 'fills' and these are minimised. The small errors remaining provide the drive to the oscillators. Since the store fill is a function of line delay as well as true phase difference, the system operating frequency is also a function of delay. In the case of the double-ended system, the error signals are more nearly the true phase differences. In this case the system frequency and the residual phase errors are nearly independent of the line delays. Suitable choice of parameter can make the frequency entirely independent of the delays.

In the correct mode of operation, the phase differences will all be close to zero so that the error signals are all small, if the oscillator controls have high gains. The magnitude of the permitted phase differences will determine a lower bound for the gains. The store fills will be affected by the delays, particularly in the double-ended system which attempts to minimise the phase difference and not the store fill. Thus the stores will have to be large enough to accommodate the expected delay variations.

If a phase comparator passes through a discontinuity in its output characteristic, it is possible for other, larger values of phase difference to generate the same apparent phase difference and hence error signal (see fig. *5.1*). The system frequency will be stable but the store fills will exceed their capacities so that traffic is lost. This will not prevent control signals from passing between the exchanges. This is 'wrong mode operation'.

To discuss the various modes of operation, an $(n - 1)$-dimensional state-space is constructed with $n - 1$ principle phase differences as the coordinates. The various comparator discontinuities will define planes, and thus distinct regions, within the state-space. It will be found that motion can occur from one region into another. Since the Laplace Transforms (Chapter 2) are only valid within a single region, the Final Value Theorem cannot be applied directly to obtain the steady state conditions. Where this theorem is applied, it is assumed that the known final region has been entered. In Chapter 4 it has already been shown that the point of entry to a region does not affect the steady state conditons within that region.

Each region will define a possible steady state point. This point may or may not lie inside the region defining it. If it does lie in the region, it gives rise to a real operating point and hence a 'wrong mode' when the point is not the origin. Points lying outside the region defining them are called virtual operating points. They have no physical reality, but are useful when discussing the behaviour of trajectories within the region. All such trajectories will leave the starting region and come under the influence of other operating points until a steady state is found. Occasionally, operating points are found that lie on the boundary of their defining region.

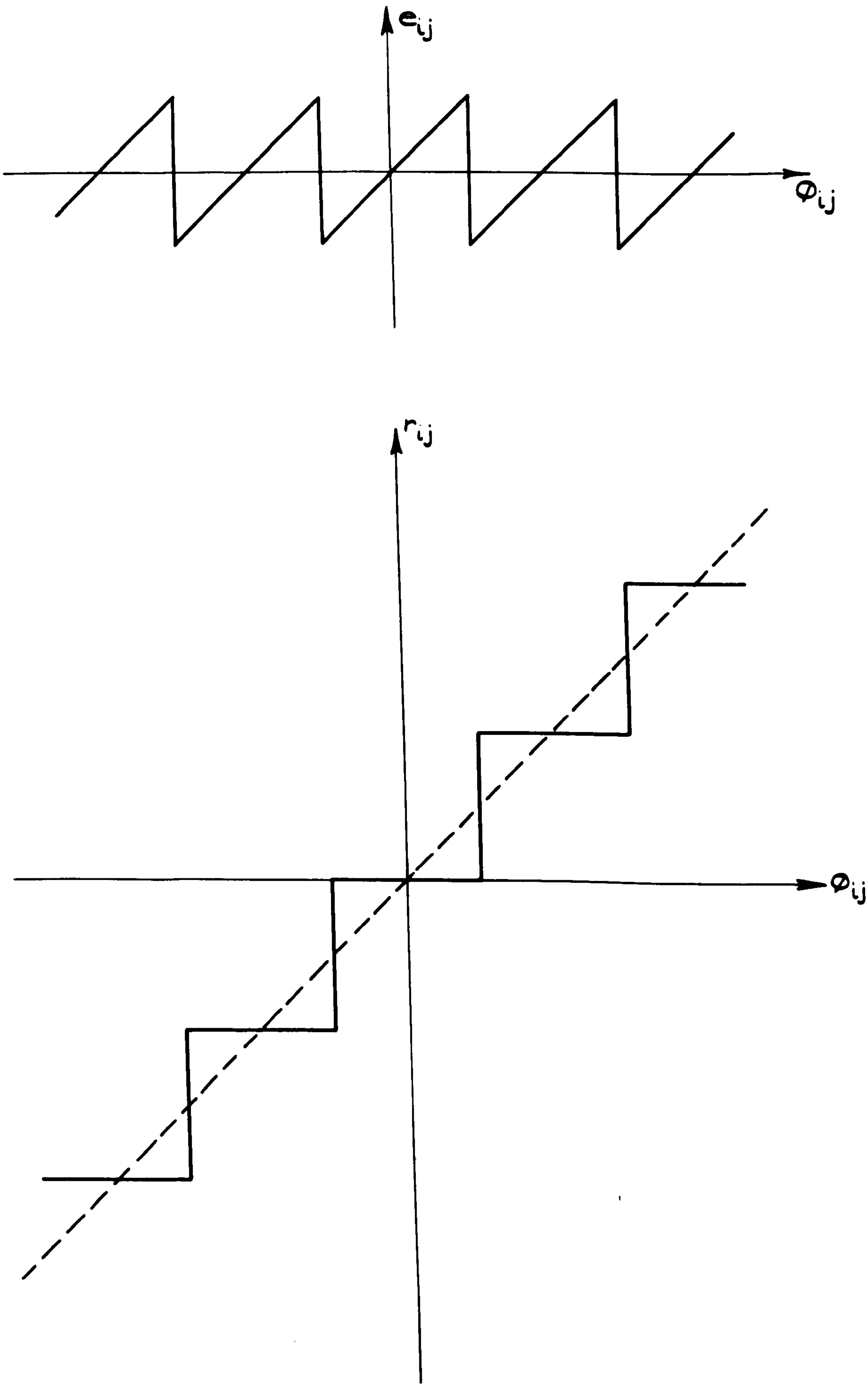


FIG. 5.1. PHASE COMPARATOR OUTPUT CHARACTERISTIC.

Provided that the stability criteria discussed in Chapter 3 are satisfied, it follows that all real operating points are stable. Virtual operating points will attract the trajectories until the boundary of the region is reached. Operating points on the boundary are quasi-stable since trajectories starting within the region and in the neighbourhood of the point will approach it as time increases, but any small disturbance can start a trajectory into an adjacent region.

In a large network there will be a great number of real stable operating points. It is the existence of these points that gives rise to the problem of wrong mode operation.

5.1 Operating mode

The operating mode is not unique and much work[15, 19, 23, 27, 29] has been devoted to this problem. The method developed by Parks and Miller is based upon a linear approximation for the system equations in the vicinity of the operating point, and makes use of a phase-space representation.[27] The results of Saitō[23] agree but are derived by a slightly different means.

It is the many possible values of the integers $r_{ij}(t)$ in the system equations which cause the wrong modes. New variables $q_{ij}(t)$, N_{ij} and δ_{ij} are defined for substitution into eqns. (*2.1.1*) and (*2.1.2*).

The integer N_{ij} and the incremental delay δ_{ij} are defined by

$$\int_{t-d_{ij}}^{t} \{f_j(\tau)\}_{G_{ij}}.d\tau = N_{ij} + \int_{t-\delta_{ij}}^{t} \{f_j(\tau)\}_{G_{ij}}.d\tau \qquad \ldots \quad (5.1.1)$$

where $-\frac{1}{2} < \int_{t-\delta_{ij}}^{t} \{f_j(\tau)\}_{G_{ij}}.d\tau < +\frac{1}{2}$

The integer $q_{ij}(t)$ is defined by

$$q_{ij}(t) = N_{ij} - r_{ij}(t) \qquad \ldots \quad (5.1.2)$$

In the steady state these relations become, respectively,

$f_s.d_{ij} = N_{ij} + f_s.\delta_{ij}$ and $q_{ij} = N_{ij} - r_{ij}$ where $-\frac{1}{2} < f_s.\delta_{ij} < +\frac{1}{2}$

In the close vicinity of the steady state points the q_{ij} and r_{ij} can usually be considered as constants. When $\delta_{ij} = 0$ and $\phi_i = \phi_j$, then $q_{ij} = 0$.

Assuming that $|\dot{\phi}_i - f_s| \ll f_s$ for all i, $\phi_j(t - d_{ij}) \rightarrow \phi_j(t) - d_{ij}.\dot{\phi}_j$. The equations for the single-ended control system (*2.1.4*) may now be written as

$$\dot{\phi}_i(t) + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} d_{ij} \dot{\phi}_j(t) = f_{oi} + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \{\phi_j(t) - \phi_i(t) + r_{ij}\}$$

$$\ldots \quad (5.1.3)$$

Putting $k_{ij} = k_i.a_{ij}$ where $a_{ij} = a_{ji} = 1$ or 0, the equations,(*5.1.3*) may be expressed in the matrix form

$$(\underline{I} + \underline{K}.\underline{D})(\dot{\underline{\phi}} - f_s.\underline{c}) = (\underline{f}_o - f_s.\underline{c}) + \underline{K}.\underline{A}.\underline{\phi} + \underline{K}.\underline{R}.\underline{c} - f_s\underline{K}.\underline{D}.\underline{c}$$

$$\ldots \quad (5.1.4)$$

Applying equations (*5.1.1*) and (*5.1.2*) to eqn. (*5.1.4*) to eliminate $\underline{R}$ and $\underline{D}$ from the right-hand side

$$(\underline{I} + \underline{K}.\underline{D})(\underline{\dot{\phi}} - f_s.\underline{c}) = (\underline{f}_o - f_s.\underline{c}) + \underline{K}(\underline{A}.\underline{\phi} - \underline{Q}.\underline{c} - f_s.\underline{\Delta}.\underline{c}) \quad \ldots \quad (5.1.5)$$

$$= \underline{0} \text{ in the steady state.}$$

The matrices $\underline{R}$, $\underline{A}$, $\underline{D}$, $\underline{Q}$, $\underline{\Delta}$ have elements r_{ij}, a_{ij}, d_{ij}, q_{ij}, δ_{ij}. The diagonal matrix $\underline{K}$ has elements k_i and all the elements of the vector $\underline{c}$ are +1. Where $a_{ij} = a_{ji} = 0$, $d_{ij} = \delta_{ij} = r_{ij} = q_{ij} = 0$ also.

All of the phase differences, as measured by the comparators, may be written in terms of the $n - 1$ phase differences $x_1 = \phi_1 - \phi_n$, $x_2 = \phi_2 - \phi_n$, ... $x_{n-1} = \phi_{n-1} - \phi_n$; these are written as the $n - 1$ dimensional column vector $\underline{x}$. In matrix form this subtraction may be expressed as $\underline{x} = \underline{B}.\underline{\phi}$, where

$$\underline{B} = \begin{bmatrix} 1 & 0 & 0 & . & . & 0 & -1 \\ 0 & 1 & 0 & . & . & 0 & -1 \\ 0 & 0 & 1 & . & . & 0 & -1 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 & -1 \end{bmatrix}$$

and $\underline{B}.\underline{c} = \underline{0}$. Other subtractions are possible, in which case the -1 elements will have different positions. There are advantages in these other definitions of $\underline{B}$ when the nth exchange is not connected to all the others. After performing this subtraction, the resulting $n - 1$ equations may be written in matrix form; the new matrices $\underline{A}^*$ and $\underline{D}^*$, defined by the subtraction, will satisfy $\underline{A}^*.\underline{B} = \underline{B}.\underline{A}$, $\underline{D}^*.\underline{B} = \underline{B}.\underline{A}$. Equations (5.1.5) now become, for $k_i = k$,

$$[\underline{I} + k.\underline{D}^*]\dot{\underline{x}} = k.\underline{A}^*.\underline{x} + \underline{B}.\underline{f}_o - k.\underline{B}\,\underline{Q} + \underline{f}_s.\underline{\Delta}\,\underline{c}$$

$$= k.\underline{A}^*.\underline{x} + \underline{B}.\underline{f}_o - kf_s.\underline{B}.\underline{\Delta}.\underline{c} - k.\underline{\lambda}$$

$$= \underline{0} \text{ in the steady state} \quad \ldots \quad (5.1.6)$$

The $n - 1$ dimensional column vector $\underline{\lambda} = \underline{B}.\underline{Q}.\underline{c}$ determines the operating mode.

Where all the $k_i \neq k$, the steady state condition is that

$$\underline{A}^{*}.\underline{x} = \underline{\lambda} + f_s.\underline{B}.\underline{\Delta}\circ\underline{c} - \underline{B}.\underline{K}^{-1}.(\underline{f}_o - f_s.\underline{c}) \qquad \ldots \quad (5.1.7)$$

This differs from the case given by eqn. (*5.1.6*) only in the effect that the initial frequencies produce on the final phase differences. The other factors affecting the steady state phase differences are the variations of the line delays from multiples of one 'frame' and the vector $\underline{\lambda}$. The first two effects will usually be small, but $\underline{\lambda}$ can have considerable effect.

Neglecting the frequency and delay effects, non-zero solutions in $\underline{x}$ are sought such that at each exchange the sum of the phase comparator outputs is zero. This is wrong mode operation since $\underline{\dot{x}} = \underline{0}$. The solution given by $\underline{\lambda} = \underline{0}$ corresponds to the correct mode of operation, which is desired in practice because the buffer store fills are minimised.

The sum of the true phase differences at each exchange will always be an integer multiple of one frame; this multiple is, from eqn. (*5.1.9*), λ_i. In any mode the sum of the true phase differences around any closed loop in the network will also be an integer multiple of one frame.

The vector $\underline{\lambda}$ will have integer entries, positive, negative or zero. In general, where $\underline{B}$ has the form given above,

$$\lambda_i = \sum_{j=1}^{n} q_{ij} - \sum_{j=1}^{n} q_{nj} \text{ where } q_{ij} = \left[\tfrac{1}{2} + x_i - x_j - f_s\delta_{ij}\right]. \qquad \ldots \quad (5.1.8)$$

Increasing any x_i in steps of 1 will cause the corresponding q_{ij} to

increase in steps of 1 and the q_{ji} to decrease in steps of 1.
By translation of the origin it may be seen that the phase space is divided into hypercubes all identical to the region $|x_i| < \frac{1}{2}$ for all $i = 1, 2, \ldots (n - 1)$. It is thus possible to consider only this region for which $|\lambda_i| < n$. When $\underline{\lambda} = 0$,

$$\lambda_i = \sum_{j=1}^{n-1} q_{ij} \text{ for which } \sum_{i=1}^{n-1} \lambda_i = 0$$

It is interesting to note that any change in δ_{ij} such that $\delta_{ij} + \delta_{ji}$ is constant displaces the operating point for which $\underline{\dot{x}} = \underline{0}$ and the basic hypercube identically, provided that only $n - 1$ such delays, linking all n stations, are subject to the change.
This fact will be referred to in Chapter 7.

The basic hypercube of the state space is intersected by hyperplanes each corresponding to the overflow of a phase comparator. The hyperplanes define regions within which all the q_{ij} are constant and thereby generate a value of $\underline{\lambda}$ peculiar to the whole region. This vector $\underline{\lambda}$ in turn defines a point $\underline{x} = \underline{A}^{*-1}.\underline{\lambda}$; this point may lie inside the region generating the $\underline{\lambda}$ in which case it defines an operating point.

Trajectories of $\underline{x}$ starting within the region move towards this operating point and will tend to it as $t \to \infty$ unless a bounding hyperplane is first encountered. If the point $\underline{A}^{*-1}.\underline{\lambda}$ does not lie in the region generating the $\underline{\lambda}$, the trajectories will leave the region. At the bounding hyperplane a different value of $\underline{\lambda}$ will apply. The trajectories will change direction and approach the newly defined point.

Curved trajectories, in a region defining a wrong mode or containing the origin, may leave the region and arrive at a different steady-state point. Thus the region of attraction of an operating point is not necessarily identical to the region defining it.

The trajectories can be investigaged by inverting the Laplace transforms obtained in Chapter 2. The resulting equations will be valid for a single region of the state-space with constant q_{ij} and hence $\underline{\lambda}$. At the boundary of the region the end conditions will become the initial conditions for the new region, apart from the change in $\underline{\lambda}$. It will be seen, however, that fully interconnected networks give rise to straight trajectories of $\underline{x}$, so that this process is unnecessary in this case.

Another possibility[23] is that no $\underline{\lambda}$, including $\underline{0}$, will generate a point lying in its corresponding region. This could occur particularly with large differences in uncontrolled frequency or with values of $|f_s \cdot \delta_{ij}| \to \frac{1}{2}$.

The general problem of finding all possible modes for a general network is unsolved, but for a particular case certain modes may readily be found. Moreover, it is impossible for wrong modes to occur in the region $|x_i| < \frac{1}{4}$, as no comparator overflows can occur in this region. This property is made use of in devices to eliminate wrong mode operation (see section 7.3).

Certain cases such as bilateral systems with $n = 4$, have some or all of the equilibrium points, other than the origin, actually on the inter-regional boundaries. Thus the trajectories from one region enter the 'equilibrium' point, at which stage a new value of $\underline{\lambda}$ is generated and the motion proceeds into the next region. These are the quasi-stable operating points.

The 'wrong mode' problem also affects double-ended systems. As $\dot{\phi}_i = \dot{f}_i \rightarrow 0$, the system equations become

$$\left\{1 - \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}(d_{ji} + \tau_{ij})\right\}\dot{\phi}_i + \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}(d_{ij} + \tau_{ij})\dot{\phi}_j$$

$$= f_{oi} + 2\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}(\phi_j - \phi_i) + \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}(r_{ij} - r_{ji}) \qquad \ldots (5.1.9)$$

In the steady state all $\dot{\phi}_i = f_s$ so that

$$2\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}(\phi_j - \phi_i) + (f_{oi} - f_s)$$

$$+ \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\left\{(r_{ij} - f_s d_{ij}) - (r_{ji} - f_s d_{ji})\right\} = 0 \qquad \ldots (5.1.10)$$

In matrix form, with $k_{ij} = k_i$,

$$2\,\underline{K}.\underline{A}.\underline{\phi} + (\underline{f}_o - f_s.\underline{c}) = \underline{K}\{\underline{Q} - \underline{Q}' + f_s(\underline{\Delta} - \underline{\Delta}')\}.c \qquad \ldots (5.1.11)$$

If $\underline{\Delta}$ is an antisymmetric matrix, then $\underline{Q}$ is also and the equation becomes

$$\underline{A}.\underline{\phi} = (\underline{Q} + f_s.\underline{\Delta}).c - \tfrac{1}{2}\underline{K}^{-1}(\underline{f}_o - f_s.\underline{c}) \qquad \ldots (5.1.12)$$

This is similar to equation (*5.1.6*) for the single-ended system. Thus the modes found for the single-ended system also apply to the double-ended system. The steady state points will be modified by the effect of the uncontrolled frequencies, which is different for the double-ended system. As the gains are reduced for stability reasons, the final effect will be almost identical.

As the delays are varied, the hyperplanes separating regions of constant $\underline{\lambda}$ will split to give additional regions. For small changes the resulting operating points are always virtual operating points. Although eqn. (*5.1.12*) will not then hold for the double-ended system, this system behaves in a similar manner to the single-ended system.

Other workers have used a different method for deriving the operating mode[39]. This is based on the concept that the reference phase may be taken as zero and the remaining phase variables used as the vector $\underline{x}$. If this is done, the entries in the connection matrix $\underline{A}$ that refer to this zero phase may be deleted. Using the nth phase as the reference, this required the deletion of the nth row and column of $\underline{A}$ to form the reduced matrix $\underline{A}^{o}$.

It may be readily demonstrated that the modes resulting from this operation are the same as those derived from the matrix $\underline{A}^{*}$. This is set out as Theorem III in section 9.2 of the Appendix. It is, of course, immaterial which of the phases is used as a reference.

The advantage of using the matrix $\underline{A}^{*}$ in preference to $\underline{A}^{o}$ is that for the fully interconnected network $\underline{A}^{*} = n\underline{I}$. For this type of network the modes may be found quite readily.

From Theorem III it is also possible to deduce an additional property of the connection matrix $\underline{A}$. This is that all the elements of the adjugate of $\underline{A}$ are equal, for all $\underline{A}$. This is proved as Theorem IV.

5.2 Modes for fully interconnected systems

The discussion of operating modes will be restricted to bilateral systems for which $k_{ij} = 0$ if $k_{ji} = 0$. When the network is fully interconnected, the matrix $\underline{A}^* = n.\underline{I}$. The trajectories are thus always straight lines to the point at which $\dot{\underline{x}} = \underline{0}$. This steady-state point is given by

$$\underline{x} = \left\{\frac{1}{n}\right\}\underline{\lambda} + \left\{\frac{f_s}{n}\right\}\underline{B}.\underline{\Delta}.\underline{c} - \left\{\frac{1}{nk}\right\}\underline{B}.\underline{f}_o \qquad \ldots \quad (5.2.1)$$

by substitution into eqn. (5.1.6).

Substituting for x_i and x_j in eqn. (5.1.8),

$$\lambda_i = \sum_{j=1}^{n-1}\left[\frac{1}{n}(\lambda_i - \lambda_j) - f_s\left\{\delta_{ij} - \frac{1}{n}\sum_{l=1}^{n}(\delta_{il} - \delta_{jl})\right\} + \frac{1}{nk}(f_{oj} - f_{oi}) + \tfrac{1}{2}\right] \qquad \ldots \quad (5.2.2)$$

All vectors $\underline{\lambda}$ which define a stable operating point must satisfy this equation.

When $n = 3$, the only possible non-zero values of $\underline{\lambda}$ are $\begin{Bmatrix}-1\\1\end{Bmatrix}$ and $\begin{Bmatrix}1\\-1\end{Bmatrix}$; the corresponding phase states are surrounded by the regions in which these values of $\underline{\lambda}$ are generated. Thus a sufficiently small variation of either $\underline{f}_o$ or of $\underline{\Delta}$ will still lead to stable states in these regions.

As $\underline{\Delta}$ is varied, additional regions of constant $\underline{\lambda}$ are produced, but these can be shown to contain no stable operating point in the case of $n = 3$ for small $f_s\delta_{ij}$. Fig. 5.2 indicates the effect of some changes in $\underline{\Delta}$ for this case. It will be seen that there are different paths for the trajectories out of these new regions, depending on the sign of the delay changes and hence on the actual value of $\underline{\lambda}$ in the new regions. The operating points are all shifted by an amount

$\left\{\frac{f_s}{n}\right\}\underline{B}.\underline{\Delta}.\underline{c}$; this is also indicated in the figure. It is also

possible for the changes in $\underline{\Delta}$ to introduce a non-zero q_{in} for some i. This will introduce additional terms to eqn. (5.2.2), as in eqn. (5.1.8).

Any variation of $\underline{f}_o$ will affect the steady-state operating point, as shown in fig. 5.3. This shift will not affect the points in the phase-space that generate particular values of $\underline{\lambda}$, but it may cause the operating point to move outside the region generating it, and eqn. (5.2.2) will be violated.

It is possible that these variations in $\underline{f}_o$ and $\underline{\Delta}$ may cause all the operating points to become unstable, including the point corresponding to $\underline{\lambda} = 0$. Under these circumstances the oscillators are outside their pull-in range. Other authors[32] have discussed this situation algebraically.

If is of interest to note that, in the absence of the above variations, the stable operating points always lie in the hyper-plane $\sum_{i=1}^{n-1} x_i = 0$. This follows from eqn. (5.2.1) and eqn. (5.1.8) for the basic hypercube. In the next section it will be found convenient to project the phase-space onto this plane, for the case of $n = 4$.

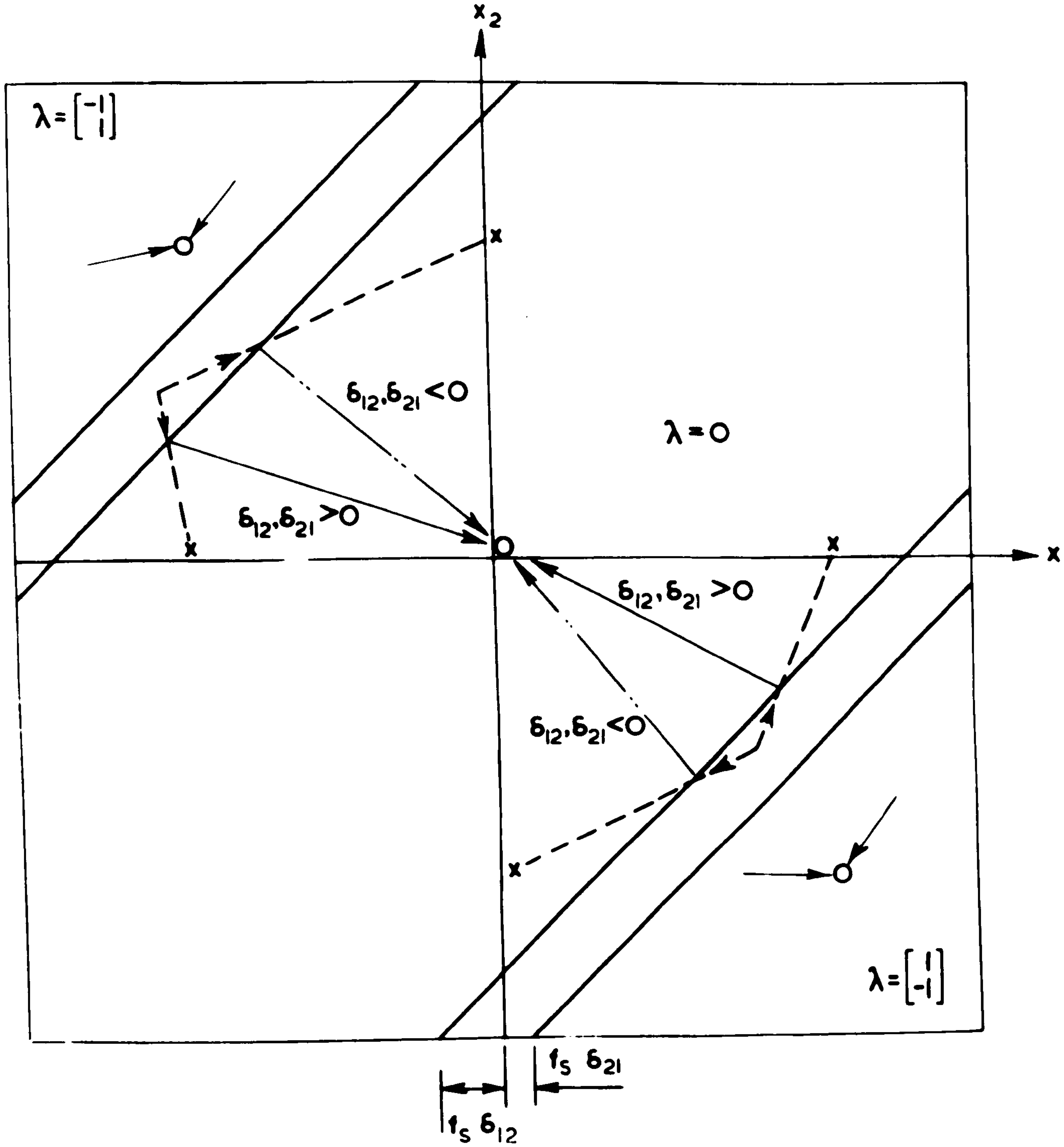


FIG. 5.2. EFFECT OF DELAY VARIATIONS ON OPERATING MODE.

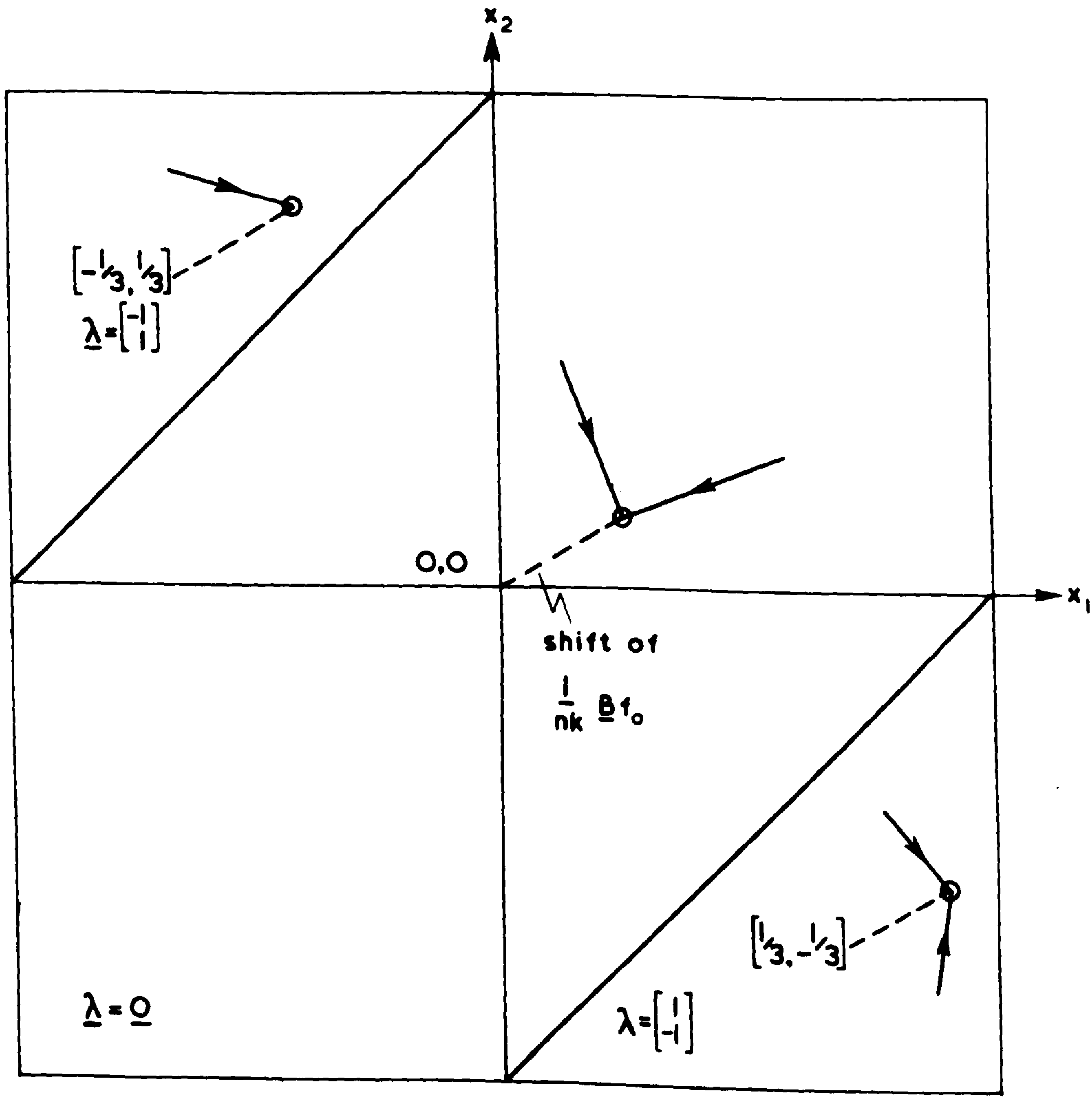


FIG. 5.3. EFFECT OF INITIAL FREQUENCY VARIATIONS ON OPERATING MODE.

In general there are det.$|\underline{A}^*| = n^{n-1}$ lattice points within the hypercube. Each is generated by a particular vector $\underline{\lambda}$, but only those lattice points in the hyperplane $\sum_{i=1}^{n-1} x_i = 0$ can generate steady steady state equilibrium points (or modes); there are $\frac{1}{n}$ det.$|\underline{A}^*| = n^{n-2}$ of these peculiar to any given hypercube. Not all of these will generate modes, as can be seen in the case of $n = 4$. In this case only 13 of the 16 possible lattice points generate a mode, and of these 12 are quasi-stable.

Some examples of the many possible modes for higher order fully-interconnected systems are all the permutations of the following vectors $\underline{\lambda}$.

$$n = 5 : \underline{\lambda} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$n = 7 : \underline{\lambda} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 3 \\ -3 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \\ -1 \\ -2 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

Where $-\underline{\lambda}$ is not a permutation of $\underline{\lambda}$, then this is also a mode. These account for 84 and 2540 of the possible 125 and 15407 lattice points respectively. The origin is also a mode and a number of other lattice points lie within the cube of $|x_i| < \frac{1}{2}$. It is likely, however, that a number of other vectors $\underline{\lambda}$ exist for these two cases.

5.3 Quasi-stable modes: four exchange systems

Saito[33] has shown that a four exchange network with one unilateral link and five bilateral links can have stable wrong modes. This type of network is unlikely to be used in practice for reasons of security against line faults. This section, therefore, considers only the case of bilateral control.

It can be shown that any ring network can have wrong modes with the exchanges separated by phase differences of $\frac{2r\pi}{n}$ for all integer values of r up to $\frac{1}{2}n$. This applies to the case of $n = 4$, for which the differences are $\frac{1}{2}\pi$ or π. In terms of the frame units used, these differences are $\frac{1}{4}$ or $\frac{1}{2}$. The latter implies that the phase comparator operates exactly on its discontinuity, which is not possible in practice. This is a quasi-stable mode. The mode with differences of $\frac{1}{4}$ is, however, stable.

For this particular network it is useful to define the phase differences as $x_i = \phi_i - \phi_{i+1}$, from which it can be seen that a cube of the phase space contains all the possible states. In this rather special case, all the modes lie on the line $x_1 = x_2 = x_3$. The cube is divided into three regions by the planes $x_1 + x_2 + x_3 = \pm\frac{1}{2}$ and the only stable modes are those for which all the x_i are $\pm\frac{1}{4}$. This example indicates that the rules for finding wrong modes described earlier in this Chapter only apply where the matrix $\underline{B}$ is as shown in section 5.1.

When a fifth link is added to the network, two exchanges are connected to all the others. One of these may be taken as reference, and the vectors $\underline{\lambda}$ found. The matrix $\underline{A}^*$ is not in this case the identity matrix $\underline{I}$. The only vector which satisfies the equation (5.1.7) is

$$\underline{\lambda} = \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix} \text{ and its permutations.}$$

The six possible phase states give four stable modes and two quasi-stable modes on the surface of the cube. These latter have four of the five phase comparators working on the discontinuity, where the output is only theoretically zero.

When the network is fully interconnected, all the modes are only quasi-stable[29, 37]. The cube of phase space defined by $|x_i| < \frac{1}{2}$ is divided by six internal planes into thirteen pieces, one of which contains the origin and the points $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. This region is that for which no phase comparator has overflowed. The other pieces are six skew pyramids containing the corners and six pentahedra. An example of each is shown in fig. 5.4, together with the trajectories for the piece. It will be seen that in the twelve regions for which overflows have occurred all the trajectories enter a point on the surface of the piece. This point is stable in the sense that trajectories within the piece approach it asymptotically, but any small disturbance can cause the trajectories to enter the origin region, or the corresponding region of an adjacent cube. Entries from adjacent cubes will occur on the faces of the origin region of the basic cube.

To clarify the inter-relation between the various quasi-stable points and the origin region, it is possible to project the cube and

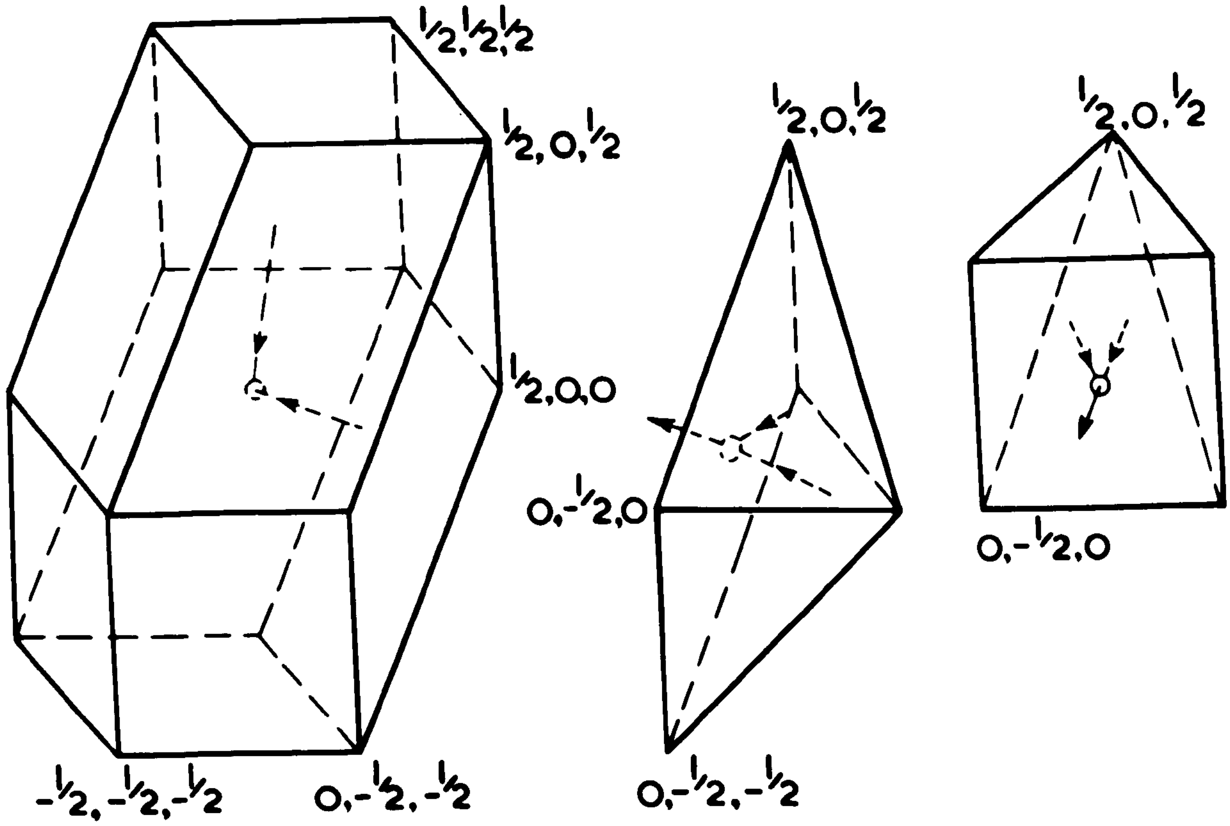


FIG. 5.4. SECTIONS OF CUBE FOR 4-EXCHANGE NETWORK.

trajectories onto the plane normal to the (111) line; this plane is defined by $x_1 + x_2 + x_3 = 0$, and all the mode points lie in it. This is shown in fig. 5.5, and trajectories that enter adjacent cubes are shown re-entering the basic cube on the opposite face.

The only stable point is the origin (0, 0, 0). The other points for which $\dot{\underline{x}}$ should be zero are the six permutations of $(\frac{1}{4}, -\frac{1}{4}, 0)$ and the six permutations of $(\pm\frac{1}{4}, \mp\frac{1}{2}, \pm\frac{1}{4})$.

For higher order fully interconnected systems where n is even, many of the possible mode points are quasi-stable points, but other modes exist which are perfectly stable.

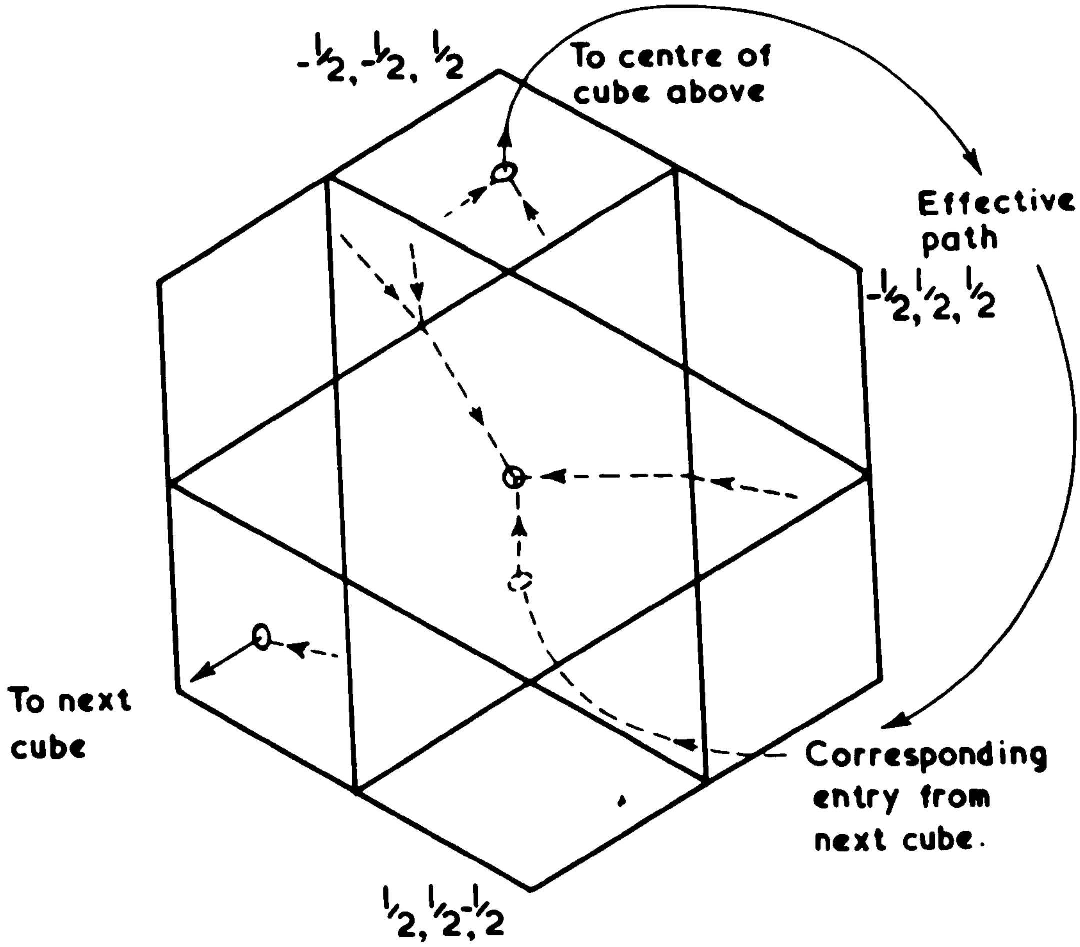


FIG. 5.5. PROJECTION OF CUBE ONTO PLANE

$x_1 + x_2 + x_3 = 0$

5.4 Blocking in non-linear systems

The problem of wrong mode operation is not restricted to linear systems. Any system with cyclic phase comparators can have values of phase differences for which there is no control action.

Duerdoth has proposed a number of systems in which the phase differences are monitored and cause a demand for control action when they exceed a certain threshold. This demand persists until a lower value of phase difference is produced. Because of the non-linear nature of the comparator outputs, these outputs are combined in a non-linear manner. Outputs can be zero, or either 'advance' or 'retard' instructions. In the presence of conflicting demands, an exchange will take no action; usually the resultant blocking will cease after a time and the phase differences will be minimised. Examination of particular cases shows that at all times, under normal conditions, at least one phase difference is being minimised.

The action of the discontinuities in the phase comparators will lead to a special form of blocking analagous to wrong mode operation. In the case of three exchanges connected in a ring network, the phase differences which occur in the wrong modes for linear systems will, in the non-linear systems, produce an 'advance' and a 'retard' signal at every exchange. This blocks all controls. In place of a region of attraction to the wrong mode, found in the linear system, there is a region in which no action takes place at all.

Higher order networks will have a large number of regions in which blocking occurs at three or more exchanges. It is possible for parts of the network to operate correctly while other small regions do not, depending on the operating mode. Proposals for avoiding this blocking are included in the system design by Duerdoth[38].

Figures 5.6 and 5.7 show phase plane diagrams for the two networks of three exchanges, with two and three links respectively. It will be seen that the hysteresis effect is neglected. The arrows show the direction of trajectories in each region. The origin region will be reached by motion in the direction of the appropriate arrow until a regional boundary is reached. The motion will then change direction and proceed in the direction of the arrows in the new region, often along the boundary between regions. Some examples are shown in the figures.

By an extension of the arguments used in Chapter 6, it is possible to show in section 6.3.3 that the non-linear system is stable in a large part of the unblocked region. Other arguments based on 'furthest advanced or retarded' exchanges have been used to the same effect.

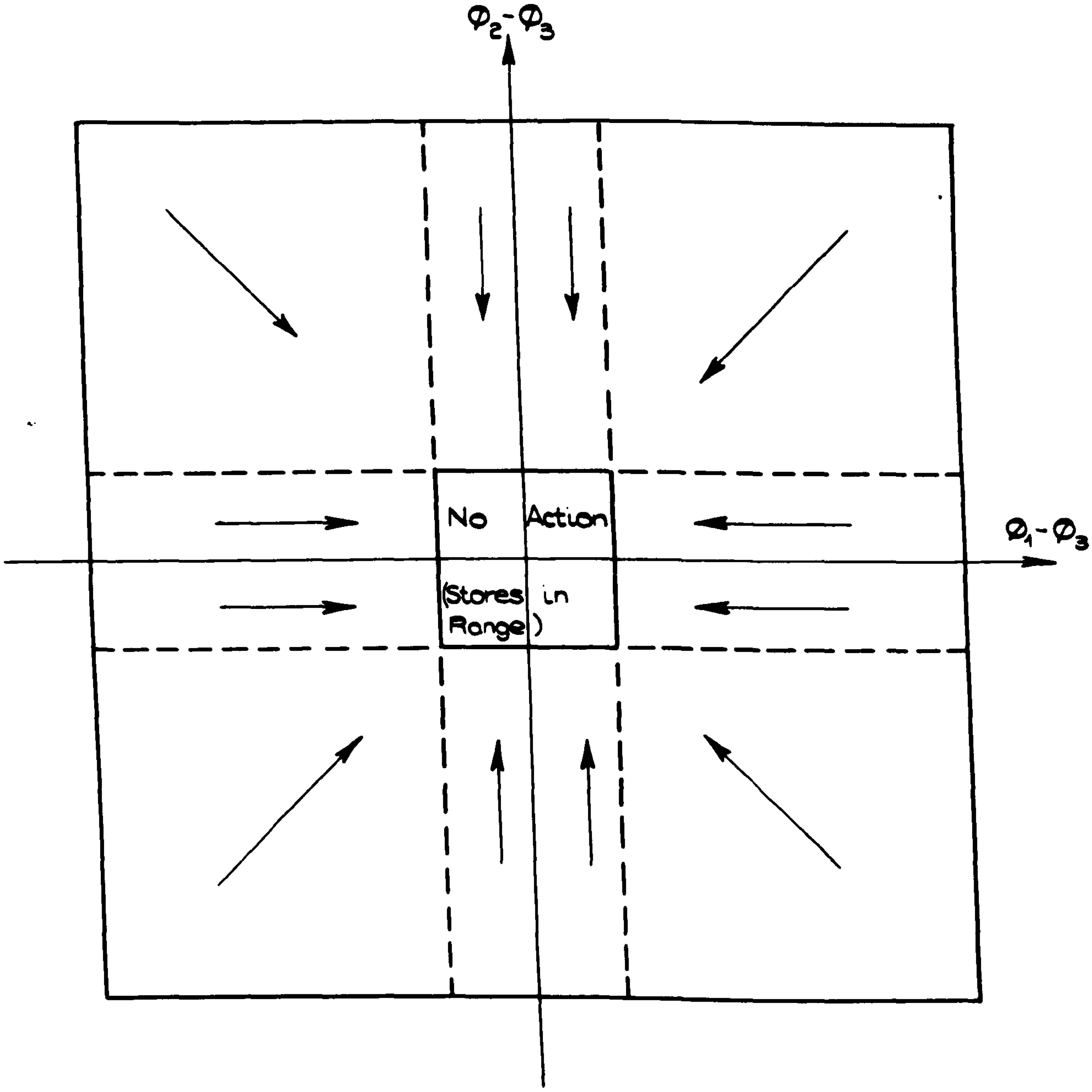


FIG. 5.6 PHASE-PLANE DIAGRAM FOR THREE EXCHANGE NON-LINEAR NETWORK.

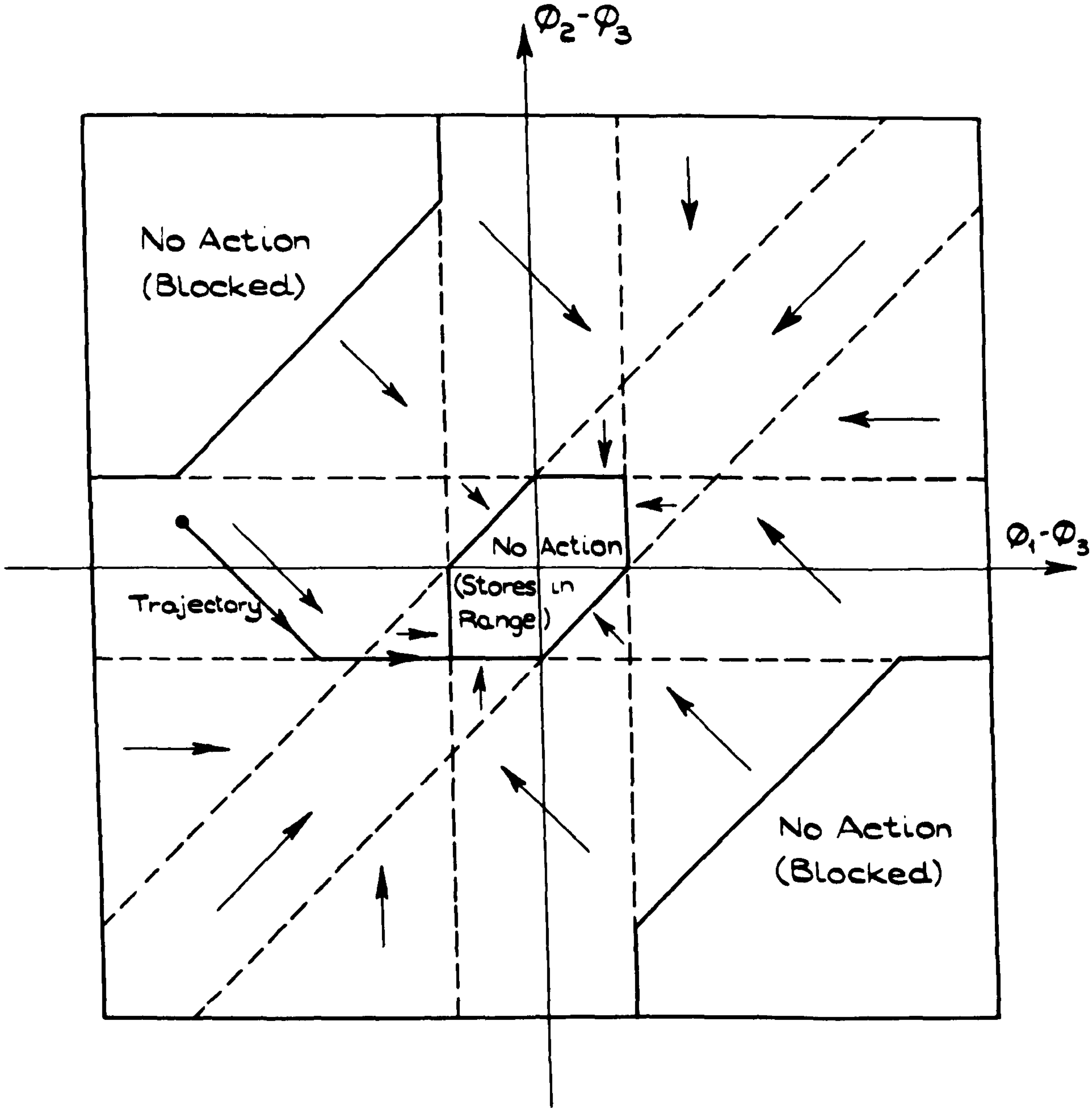


FIG. 5.7 BLOCKING IN THREE EXCHANGE NON-LINEAR NETWORK

6 NOISE AND TRANSIENTS

A basic problem in the design of a control system is that of noise. Many sources of noise exist, but little is as yet known of the spectral densities at frequencies likely to affect the control system. For this reason it is not possible to calculate explicitly the effects of the noise on the phase differences. It is therefore essential that experimental work be devoted to this aspect of the problem. This should be one of the tasks of a field trial.

Examination of the closed loop response of the systems indicates that any noise present will be filtered by the control system. Additional filters will increase the attenuation at all but the lowest frequencies. It is at these frequencies, therefore, that measurements of noise must be made.

It is possible to estimate the maximum transients that can occur in the system after it is released from some initial state. This is possible since certain functions of the state variables may be used as Liapunov functions. This suggests a technique for the extension of the network.

6.1 Noise sources

In addition to amplifier noise and other common disturbances in control systems, there are sources of noise peculiar to the digital communications environment. Some of these are discussed below.

6.1.1 Jitter

The digital repeaters are designed to recover the originating clock rate from the message stream and to generate an internal clock at this frequency. The outgoing digital message is obtained by gating the incoming pulses with the internal clock pulses. The accuracy with which the repeater follows the originating clock frequency is dependent on the structure of the binary code of the message. Certain message patterns allow good accuracy, while others permit the timing to 'wander' within certain limits. The outgoing message is thus subject to timing 'jitter'.

As more repeaters are added, each will increase the 'jitter' in the timing until the message becomes mutilated. This sets a limit to the length of a p.c.m. system.

At a digital exchange, the message will be completely retimed by the clock at that exchange. However, the jitter will be added to the phase difference signal used as an error signal to control the exchange clock frequency. The jitter will thus be present in the timing of the output of the digital exchange. It cannot cause instability, but it will increase the amount of buffer storage required at every input to every exchange in the network. An estimate of the mean square of the noise at each exchange will permit the correct store size to be chosen.

6.1.2 Sampling

The jitter noise spectrum is known and is mainly concentrated in the region of the bit-rate used in the digital transmission system. This noise will be added to the signals of the buffer store fills and thus introduced into the control system.

The buffer store fill signal is sampled at a frequency not greater than the slot rate of 192 kHz. In some designs the store fill is only obtained at greater intervals of time, and the sampling rate is correspondingly reduced. The jitter spectrum is thus sampled at the rate of 192 kHz or less, and all the jitter power will be concentrated into an effective bandwidth of one half the sampling rate. It is not, of course, possible to filter the jitter noise before the sampling takes place. The resulting noise spectrum is in all probability 'white', but this needs to be investigated. After the buffer store fill is determined, it is possible to filter the signals and reduce the high frequency amplitudes. For best results, the smapling should be at the highest possible frequency, thus reducing the power in the low frequency part of the spectrum.

A suggested technique for designing the buffer store is to sample at the slot rate by comparison of digit pulse positions within incoming and exchange slots. This signal does not distinguish between slots and is thus limited to $\pm\frac{1}{2}$ slot. To allow for possibly larger phase differences than $\frac{1}{2}$ slot, the timing of incoming slots should be compared with that of the exchange slots. The first signals are derived at the slot rate of 192 kHz, while the second are derived at the frame rate of only 8 kHz. By a suitable comparison of the two outputs it is possible to obtain an effective sampling rate of 192 kHz without obscuring phase differences of more than one slot.

The resulting noise power at any frequency, due to jitter, will be reduced to about $\frac{1}{24}$ of that obtained with the slower rate of sampling.

6.1.3 Quantisation

The error signals are likely to be quantised within the control system. One possible application of quantisation is in the provision of the data link for the double-ended system. The non-linear systems are also examples of quantised systems.

As soon as quantised error signals are present in the system, the range of possible system frequencies becomes restricted to a number of fixed frequencies, any one of which could be the operating frequency for the whole network. Unless the predicted operating frequency is one of these fixed frequencies, errors will increase until the next step of error quantisation is reached. This will cause a step to a new operating frequency and the phase differences will start to change in a new direction. Eventually the error signals will reach their former quantised value and the operating frequency will revert to its earlier value. The system will thus operate at a frequency that switches from one value to another, but has the mean predicted by the equations in Chapter 4.

A small quantising step will cause the switching to be more rapid than with a larger step, but the level of disturbance will be small. Larger steps will require corresponding increases in the buffer store capacity. In general, this increase will be two steps of phase difference quantisation, so efforts will be made to keep this effect as small as possible.

6.1.4 Starting the network

By far the largest disturbances will occur when the network is initially connected to the control system, and when new exchanges are added to it. There is likely to be an increase in store fill at every exchange in the network after a step of this type is applied to any one point. Limits are indicated in section 6.3.

To minimise the disturbance to the network as a whole, it is suggested that an exchange should be added to an existing network as a 'slave' of some 'parent'. When the phase difference measured at the parent has diminished to a small value, that and other connected exchanges can use the signals from the new exchange to control their own clocks. It is suggested that a sequential method should be adopted for starting the network. This would be compatible with the method outlined in section 7.3 for the elimination of wrong modes, and would prevent the network from starting in such a wrong mode.

6.2 Closed loop response

An investigation of the closed loop response of a fully connected system without delays or filters is instructive in that it shows the filtering effect of the feedback loops. This applies to single- or double-ended systems.

With the noise input, the system equations (*2.1.4*) may be regarded as

$$\underline{f}(t) = \dot{\underline{\phi}}(t) = \underline{f}_o + k\underline{n}(t) = k\underline{A}.\underline{\phi}(t) \qquad \ldots \quad (6.2.1)$$

where $\underline{n}(t)$ is the vector of noise voltages. Taking Laplace transforms, this yields

$$\overline{\underline{f}}(s) = \{s\underline{I} - k\underline{A}\}^{-1}.\{\underline{f}_o + s\underline{\phi}_o + ks\overline{\underline{n}}(s)\} \qquad \ldots \quad (6.2.2)$$

The value of the *i*th component of frequency may be obtained from

$$\overline{f}_i(s) = \frac{1}{s + nk}\{f_{oi} + s.\phi_{oi} + ks.\overline{n}_i(s)\} + \frac{1}{s(s+nk)} \sum_{j=1}^{n} \{f_{oj} + s.\phi_{oj} + ks.\overline{n}_j(s)\} \qquad \ldots \quad (6.2.3)$$

$$= \frac{1}{ns} \sum_{j=1}^{n} f_{oj} + s.\phi_{oj} + ks.\overline{n}_j(s) + \frac{1}{s + nk}\left[\{f_{oi} + s.\phi_{oi} + ks.\overline{n}_i(s)\} - \frac{1}{n}\sum_{j=1}^{n}\{f_{oj} + s.\phi_{oj} + ks.\overline{n}_j(s)\}\right]$$

In the steady-state, the frequency will be the final frequency f_s, perturbed by the noise signals.

In this fully interconnected system, the values of the phase differences are possibly more instructive. It can be seen that, by subtraction of equations (6.2.3)

$$\bar{\phi}_i(s) - \bar{\phi}_j(s) = \frac{1}{s(s + nk)}\left\{f_{oi} - f_{oj}\right\}$$

$$+ \frac{1}{s + nk}\left\{\phi_{oi} - \phi_{oj} + k\bar{n}_i(s) - k\bar{n}_j(s)\right\} \quad \ldots \quad (6.2.4)$$

whence

$$(\phi_i - \phi_j)_t = (\phi_i - \phi_j)_0 . e^{-nkt} + \frac{1}{nk}(f_{oi} - f_{oj})(1 - e^{-nkt})$$

$$+ L^{-1}\left[\frac{k}{s + nk}\{\bar{n}_i(s) - \bar{n}_j(s)\}\right] \quad \ldots \quad (6.2.5)$$

The phase differences thus change exponentially from their initial values to those required to maintain the final frequency, and are perturbed throughout by the noise signals, filtered by the feedback loop. It will be noted that as the noise inputs will be independent, their effects add, although the independent random voltages are subtracted. If the inputs are of 'white' noise of the same amplitude, the combined amplitude will be $\sqrt{2}$ times the individual amplitudes. The equivalent filter has a low pass characteristic, a gain of $\frac{1}{n}$ and time constant $\frac{1}{nk}$.

To eliminate the greatest possible range of frequencies, a small value of nk is desirable. This implies an upper bound on the gains. However, the fast minimisation of initial phase differences implies a lower bound on the gains. Some compromise is essential, and this is discussed in section 7.2.

The presence of a filter in the error path will reduce the magnitude of the noise voltages applied to the oscillator, but will increase the time required to eliminate transients. It will also create a possible stability problem, so that the gains have to be reduced as already discussed in Chapter 3.

6.3 Estimation of transients

When the system is started from its initial state, unless the control network is fully interconnected without delays or filters, some phase differences will overshoot their steady state values. For phase differences that are increasing their magnitudes, this may involve an excursion such that one or more buffer stores overflow. This does not imply that the phase comparators overflow. For certain large initial phase differences, not themselves causing comparator overflow, it is possible for the subsequent motion to cause an overflow. It is thus essential that some estimate of the maximum transients be found.

The analysis of fully interconnected systems in Chapter 5 and in the previous section have shown that such systems have a monotonic response to any transient disturbance. It is now intended to obtain estimates for other classes of network.

6.3.1 Liapunov functions

The use of Liapunov functions is usually associated with the determination of stability criteria for multivariable systems, often including non-linearities. These functions may also be used to obtain estimates of the bounds on transient motion.

One method[17] is to set up a Liapunov function in the state space of frequencies from which it may be deduced that the frequency vector lies within some shrinking surface in the n-dimensional space. The modulus of frequency difference may be shown to be related to the instantaneous size of this shrinking space. It is then possible to integrate this modulus to obtain a bound on the integral of frequency difference. Since the instantaneous value of the phase difference is equal to the initial value of the phase difference plus the integral of the frequency difference, a bound on the integral implies a bound on the phase difference.

Another method is to form a set of phase differences as in Chapter 5 and to use the state space of phase differences to form the Liapunov function. This is the method used in section 6.3.3 to discuss non-linear systems. It is also applicable to the linear system with no delays. The phase difference vector is then shown to lie within a shrinking surface in the phase space. The initial phase differences determine the initial size of the surface, and subsequent magnitudes will depend upon the system time-constants. The longest time-constant is the reciprocal of the smallest non-zero latent root of the matrix $-\underline{A}$; this is also the smallest latent root of the matrices $-\underline{A}^*$ and $-A^0$. This may be derived by the Gershgorin Circle Theorem used in Chapter 3 (see section 3.1.1), but only for highly connected networks in which every exchange is connected to at least $\frac{1}{2}n - 1$ of the others. The result is the same as for case (a) of the next section (6.3.2).

In both cases it is possible to use a quadratic function of the form $\underline{x}'\underline{x}$ as the Liapunov function. This shows that the upper bound for all phase differences $x_i(t)$ is the initial value of the Euclidean state-space norm defined by $\rho^2(t) = \underline{x}'(t)\ \underline{x}(t)$. If the hypersphere defined by this function lies entirely within the region in which no phase comparator has overflowed, then it is not possible for the system to enter a wrong mode.

The lower bound on system gains discussed in section 7.2 is derived by considering initial states within the largest hypersphere in the phase-difference space and determining the longest permissable time for the phase differences to be such that the buffer stores are in a state of overflow.

6.3.2 Linear system without delays

The system equations for a network without delays or filters may be expressed as

$$\underline{\dot{\phi}} = \underline{f}_o + k\underline{A}.\underline{\phi} \qquad \ldots \quad (6.3.1)$$

where $a_{ij} = a_{ji} = 1$ or 0 and $a_{ii} = -\sum_{\substack{j=1 \\ \neq i}}^{n} a_{ij}$.

The presence of the constant terms introduced by $\underline{f}_o$ will be seen to be a complicating factor. By differentiating the equations, it is possible to examine the alternative set

$$\underline{\dot{f}} = k\underline{A}.\underline{f} \qquad \ldots \quad (6.3.2)$$

In the steady state, all the frequencies approach a common value f_s. By subtracting this from all the frequency components, and defining $y_i = f_i - f_s$, the equations (6.3.2) may be written as

$$\underline{\dot{y}} = k\underline{A}.\underline{y} \qquad \ldots \quad (6.3.3)$$

Now by addition of the n equations, $\sum_{i=1}^{n} \dot{y}_i = 0$ so that $\sum_{i=1}^{n} y_i = \text{const.}$ Since as $t \to \infty$, $y_i \to 0$, this constant is zero.

A suitable Liapunov function for the system is $V = \underline{y}'\underline{y}$, for which $\frac{dV}{dt} = 2\underline{y}'\underline{A}.\underline{y}$. If it can be shown that $\frac{dV}{dt} \leq -c\ \underline{y}'\underline{y}$, it is clear that $\rho(t) \leq \rho(0)e^{-\frac{1}{2}ct}$ where $\rho^2(t) = \underline{y}'(t)\ \underline{y}(t)$ and $\rho(0)$ is determined by the initial frequencies. Unfortunately, as it stands, $\underline{A}$ is a negative semi-definite matrix, so it is necessary to add to dV/dt the additional quadratic form $-2\mu^2 k(y_1 + y_2 + y_3 + \ldots + y_n)^2$ which is of zero value. Thus

$$\frac{dV}{dt} = -2k\{\underline{y}'(-\underline{A})\underline{y} + \mu^2\ \underline{y}'\underline{B}\underline{y}\} \qquad \ldots \quad (6.3.4)$$

where $b_{ij} = 1$, all i, j. The matrix $\left[\mu^2\ \underline{B} - \underline{A}\right]$ is clearly negative definite when $\mu = 1$.

It is now possible to estimate the constant c for various classes of network. It is also necessary to relate this to the integral of frequency difference. This is done by considering the following equalities and inequalities

$$\left| \int_0^t (f_i - f_j)_t . dt \right| = \left| \int_0^t (y_i - y_j)_t . dt \right|$$

$$\leq \int_0^t \left| y_i - y_j \right|_t . dt$$

$$\leq \int_0^\infty \left| y_i - y_j \right|_t . dt$$

$$\leq \int_0^\infty 2\rho(t) dt$$

$$\leq \int_0^\infty 2Re^{-qt} dt = \frac{2R}{q}$$

$$\ldots \quad (6.3.5)$$

The estimate of the bound is $\frac{2R}{q}$ where $R = \rho(0)$ and $q = \frac{1}{2}c$. For well connected networks this may be a smaller bound than that obtained from the consideration of the phase difference equations, which is $\rho(0)$.

Two classes of network may be distinguished. These are

(a) highly connected networks, in which each station is connected to m_i other stations, where $m_i \geq \frac{1}{2}n - 1$

(b) sparsely connected networks where (a) is not satisfied.

In case (a), the Gershgorin theorem[10] is applied to the matrix $[\mu^2\underline{B} - \underline{A}]$, where $\mu = 1$ to yield the estimate

$$c = 2k\{2(m_i + 1) - n\}_{\min i} \text{ for } i = 1, 2, \ldots n \qquad \ldots \quad (6.3.6)$$

Case (b) may be treated by considering a chain of exchanges ordered 1, 2, 3, ... n, successively. If the network contains such a chain, then

$$\frac{dV}{dt} \leq -2k \left\{ \sum_{i=1}^{n-1} (y_i - y_{i-1})^2 \right\}$$

to which may be added, as before, $-2\mu^2k(y_1 + y_2 + \ldots + y_n)^2$. This yields

$$\frac{dV}{dt} \leq -2k\ \underline{y}'\underline{T}'\underline{T}\ \underline{y} \quad \text{where } \underline{T} = \begin{bmatrix} 1 & -1 & 0 & 0 & . & . & 0 \\ 0 & 1 & -1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 1 & -1 \\ \mu & \mu & \mu & \mu & . & \mu & \mu \end{bmatrix} \quad (n \times n)$$

The required value of c is 2k times the smallest latent root of $\underline{T}'\underline{T}$, which must be determined. For non-zero μ the matrix $\underline{T}$ is non-singular and the latent roots of $\underline{T}'\underline{T}$ are equal to the latent roots of $\underline{T}\,\underline{T}'$.

$$\text{Now } \underline{T}\,\underline{T}' = \begin{bmatrix} 2 & -1 & 0 & . & . & 0 \\ -1 & 2 & -1 & . & . & 0 \\ 0 & -1 & 2 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & 2 & -1 & 0 \\ . & . & . & -1 & 2 & 0 \\ 0 & 0 & . & 0 & 0 & n\mu^2 \end{bmatrix} \quad (n \times n)$$

of which the latent roots are $n\mu^2$ and $4\sin^2(r\pi/2n)$ for $r = 1, 2, \ldots n-1$. Provided that μ is chosen such that $n\mu^2 > 4\sin^2(\pi/2n)$, the estimate is given by

$$c = 8k\sin^2(\pi/2n) \qquad \ldots \quad (6.3.7)$$

Spurs may be dealt with by counting stations twice to form a chain of length greater than n stations. The estimate will be increased by

(i) the effect on both R and c of the additional stations

(ii) the effect on R of the largest number of times a given frequency is used as a state variable.

6.3.3 Liapunov functions for non-linear systems

It is also possible to construct Liapunov functions for non-linear systems. The equations for such a system without delays or filters may be written as

$$f_i(t) = f_{oi} + k \sum_{\substack{j=1 \\ \neq i}}^{n} a_{ij}\sigma\{x_j(t) - x_i(t) - r_{ij}(t)\} \qquad \ldots \quad (6.3.8)$$

where $x_i(t) = \phi_j(t) - \phi_i(t)$ and $\sigma\{x\}$ is any function of x such that when $x < 0$, $\sigma\{x\} \leq 0$ and when $x > 0$, $\sigma\{x\} \geq 0$. For the special case where $\sigma\{x\} = x$, this equation reduced to eqn. (2.1.4) with no delays.

For such a system it is possible to use the Liapunov function $V = \underline{x}'\underline{x}$ for which

$$\frac{dV}{dt} = 2 \sum_{i=1}^{n-1} x_i . \dot{x}_i$$

Displacing the origin by an amount $\underline{B}.\underline{f}_o$ as in section 5.1, this becomes

$$\frac{dV}{dt} = 2k\ \underline{x}'\ \underline{\sigma}\{\underline{A}^*\ \underline{x}\} \text{ or } 2k\ \underline{x}'\ \underline{\sigma}\{\underline{A}^0\ \underline{x}\}$$

$$= 2k \sum_{i=1}^{n-1} x_i \sum_{\substack{j=1 \\ \neq i}}^{n-1} a_{ij}.\sigma\{x_j - x_i\}$$

in the region for which $r_{ij} = 0$.

Now the non-linearity is assumed to be symmetric so that $\sigma\{x\} = -\sigma\{-x\}$. The derivative now becomes

$$\frac{dV}{dt} = \sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ \neq i}}^{n-1} k.a_{ij}(x_i - x_j).\sigma\{x_j - x_i\} \qquad \ldots \quad (6.3.9)$$

Thus $\frac{dV}{dt} \leq 0$, and the system is stable; the equality occurs when $\sigma\{x_j - x_i\}$ is zero. The vector $\underline{x}(t)$ will lie inside the shrinking hypersphere defined by V(t), which has initial radius $\frac{1}{4}\sqrt{2}$; $V(0) \leq \frac{1}{8}$, if the whole hypersphere lies in the region for which $r_{ij} = 0$.

The basic system proposed by Duerdoth[6] uses non-linear functions with hysteresis and varying threshold values. These may be regarded as non-linearities of the on-off type within a feedback loop with particular levels of gain to determine the size of the hysteresis backlash. He also proposed a non-linear combining law. This may be regarded as a variation of the gain k between different zones of the phase space; k is always positive, so V(t) always decreases. Thus for any starting point in the hypersphere the phase differences tend to zero as time increases.

The boundary of the region of the phase-space for which all $r_{ij} = 0$ is given by

$$\sum_{i=1}^{n-1} |x_i| + \sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ \neq i}}^{n-1} a_{ij} \, |x_j - x_i| = \tfrac{1}{2}(n - 1) \qquad \ldots \quad (6.3.10)$$

Since $\frac{d}{dt}|x| = \dot{x}.\mathrm{sgn}(x)$, it would be possible to use the Liapunov function

$$V = \sum_{i=1}^{n-1} |x_i| + \sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ \neq i}}^{n-1} a_{ij} \, |x_j - x_i| \qquad \ldots \quad (6.3.11)$$

In all cases of fully interconnected networks it is clear that

$\frac{dV}{dt} < 0$ for non-linear systems, since $\underline{A}^* \equiv \underline{I}$ and hence $\dot{x}_i = -nk.\sigma\{x_i\}$. It is not clear that this function decays for all types of network, as the $\dot{x}_i$ are then functions of all the x_j as well as of x_i. It would indeed seem probable that the Liapunov function defined by eqn. (*6.3.11*) is not always stable, since sparsely connected networks can cause some phase differences to increase initially.

6.3.4 Delays

The inclusion of the effects of delays in the estimates of transients requires the use of Liapunov functionals. These have been the objects of several recent studies in the field of Automatic Control, and it is suggested that the results of such work could be applied to this problem.

6.4 Filters

The error signals will have a bandwidth of half the sampling rate used in the phase comparators. It is necessary to limit this bandwidth by means of filters, in order to limit the power applied to the oscillator control input. A simple low-pass filter is probably sufficient.

The simplest form of low-pass filter is the resistance-capacitance filter. If the control system is designed on a sampled digital basis, this type of filter is not likely to be suitable. A simple digital filter can be built, using modern techniques, in little more space than that needed for the resistor and capacitor. The same arrangement would be suitable for the data-link for the double-ended system.

The effect of the filters on the noise and transient responses is not established. This should follow a study of the noise spectrum, which will probably be a part of a field trial. The low-pass filter in the error path will attenuate the noise at all frequencies provided that the stability criteria are satisfied. The likely pass band is the range from D.C. to 100 Hz. A preliminary study of noise waveforms shows that there is little power contained in the band below 1000 Hz.

The effect of continuous filters on the continuous systems has been discussed in Chapters 3 and 4. The use of digital filters with the sampled systems will necessitate a reappraisal of the factors governing the stability of the control system. It will also be necessary to determine the steady-state frequency of the system in this case. Examination of this type of system is recommended as a subject for further study.

In a fully digital system it is recommended that the phase comparators be sampled at the highest possible frequency, 192 kHz in the case of the 24 channel system. A digital filter at this stage can eliminate most of the noise power. The filter output may then be sampled at a lower rate, preferably a sub-multiple of 192 kHz, and transmitted over the data-link to the remote exchange. The combination of local and remote phase differences can take place before filtering. The use of a filter at the oscillator input is optional in this case, as all the information is digital. The zero-order hold at this point will smooth the output sufficiently to allow its application to the oscillator.

7 SYSTEM DESIGN AND SUGGESTED FURTHER WORK

From the results given in this Thesis it is possible to establish general rules for the design of a control system for p.c.m. oscillator synchronisation. These rules may then be tested in a field trial experiment. The field trial will yield much information of interest about the environment of the control system. It will also indicate the safety margins available with the stability criteria used. Some suggestions for experiments to be included in the field trial are given at the end of this Chapter.

7.1 Initial setting of delays

Digital information sent by an exchange will normally be processed without more than a small delay by the receiving exchange. The time slot used to transmit the outgoing signals will usually be that used to transmit the return speech signals from the receiving exchange. Unless additional stores are provided at the originating exchange, the return information will not, in general, occur at the same moment as the outgoing time slot. If the total delay around the loop of 'go' and 'return' paths is an integer multiple of one frame period, the return information will be synchronised with the outgoing information for the same speech channel. Some variation of the delays is inevitable, so buffer stores are inserted as part of the total delay.

The requirement that all the possible loops in the network have total delays which are integer multiples of one frame period is a consequence of a particular exchange design. The simplest way in which this may be satisfied is to pad out all the delays so that the delay of each direction of transmission in every link is an integer multiple of one frame period. The manner in which this is to be done is not immediately obvious, and at first sight it seems that some absolute measurement of delay is required. This would be very difficult, if not impossible, to make.

Fortunately, in a network of n exchanges there is some arbitrariness in defining phases and phase differences. Phase differences for the whole network are uniquely defined once a given set of $n - 1$ phase differences, involving all n exchanges, is known. Such a set is used in the discussion of operating mode, namely $\phi_i - \phi_n$ for $i = 1, 2, \ldots n - 1$.

This fact has a direct bearing on the padding-out problem. It is clearly possible to measure the delay around every closed loop in the network. This yields a set of equations from which it is possible to determine all the unidirectional delays d_{ij} in terms of $n - 1$ basic delays. These must be chosen to be one or other of the unidirectional delays of each of $n - 1$ basic links which join all of the n exchanges. The absolute values of these $n - 1$ basic delays will affect only the $n - 1$ arbitrary phase differences, which are thus fixed.

The procedure for setting the delays is thus quite simple. First, the total delay around some loop linking two exchanges is measured and padded out by means of a single adjustment to equal an integer multiple of one frame period. It may be found convenient to attempt to equalise the delays of the two unidirectional paths.

This can be done by calculating the theoretical delay of a cable of the appropriate length, and sharing the adjustment required between two pad delays, one at each incoming line terminal. The phase of one exchange is then adjusted so that it is the same as the phase of the other, as received over the digital link. Both exchanges will now be in phase with the apparent phase of the other. The buffer stores may now be added to the lines. A measurement of the ambient temperature will indicate any necessary temperature compensation required; otherwise the buffer stores will replace a fixed delay of half the store capacity. The stores will now be half full at the mean annual temperature of the environment, while the two exchange clock frequencies remain identical.

Additional exchanges may be linked to one of the exchanges in the existing network by this means. They are linked to the remaining exchanges by adjustment of the line delay pads in each unidirectional path so that the buffer store fills are correct for the ambient temperature. As soon as all the delays are correctly set up, the exchange clock frequency, which would initially be locked to one of the other clocks, is allowed to influence the frequencies of the existing exchange clocks.

Saito[23] states that $n - 1$ delays are arbitrary but does not specify that they must be chosen to link all n exchanges.

The arbitrariness of the delays and phase-differences has no real effect on the system operating mode. The state-space diagram of hypercubes and operating points is translated in its entirety, relative to the true phase-difference axes, by an unknown quantity. This quantity could only be determined if the $n - 1$ basic delays were known. The effect of varying the line delays was indicated in fig. 5.2. If the delays in the basic links are varied, it is the increase in the loop delays which will cause the boundaries of the basic hypercube to split. The external boundaries are unchanged if the loop delays are constant. If the delays of other links are changed, the bounding hyperplanes between the regions will move while the loop delay is constant and split if the loop delay is changed. It is thus only the non-basic delays which affect the operating mode.

7.2 Choice of gains and size of buffer stores

Once the delays have been correctly set up a careful choice of the gains k_{ij} has to be made to optimise the performance of the control system.

The factors affecting the choice of gain will indicate the design sequence. These are

(1) There is an upper bound to the k_{ij} given by the stability criteria.

(2) There is a lower bound to the k_{ij} as the vector of steady state phase differences includes a term which, for a fully interconnected system, is $^1/nk\ \underline{B}.\underline{f}_0$ and is always inversely proportional to the gains. The steady state phase differences must be within the capacity of the buffer stores.

(3) There is a further lower bound as the speed of response to transients is inversely proportional to gain. The time taken to eliminate a step error must not be excessive, and should be less than that which causes the switches in the exchange to release.

(4) There is an upper bound as the bandwidth of the system increases with increasing gain, and the effects of noise must be minimised.

Of these, only the first bound is an absolute limit. All the others depend upon the size of the buffer store. The buffer store must be large enough to accommodate phase differences due to noise, to the difference in initial frequencies, and due to the temperature dependent delays. If the store size is too small it is possible for transients to occur that cause a loss of traffic for an appreciable time, due to slow minimisation of transients. Of these effects, only the temperature dependent variations are unaffected by gain changes.

In designing a control system for a particular network, the following procedure may be adopted

(i) From a study of the particular network, but with zero line delays, the k_{ij} must be chosen large enough for an equilibrium frequency to exist, given the likely magnitudes of the components of the vector of uncontrolled frequency differences, $\underline{B}.\underline{f}_0$. In fig. 5.3 this corresponds to a choice of k sufficiently large to keep the shift in the $\underline{\lambda} = \underline{0}$ operating point within the $\underline{\lambda} = \underline{0}$ region.

(ii) The $n - 1$ dimensional diagram and the lower bound on the k_{ij} will have to be modified to account for time delays as in fig. 5.2. The $f_s.\delta_{ij}$ effect on the diagram will be small as a result of the padding-out process.

(iii) The upper bound on the k_{ij} due to stability considerations must be found. The sufficient conditions given in this Thesis are at least adequate for initial design purposes. Once stability is assured, the final frequency f_s may be found from the appropriate formula; f_s must lie in the controllable range of each oscillator.

(iv) An essential part of the design process is the minimisation of the buffer store capacity. This capacity must be sufficient to deal with the small and slowly changing $f_s.\delta_{ij}$. Further

store capacity must be available to deal with the steady state phase differences. Since this capacity increase is likely to be of the order of only ¼ slot, the lower bound deduced in *(i)* above will need to be increased by a factor of 96 in a 24 channel system.

(v) Yet another increase in store capacity will be needed to deal with random phase variations due to noise. Given the noise power spectrum, it is possible to calculate the size of this increase for various values of the k_{ij}. It may be necessary to reduce the upper bound on the k_{ij} from that given in *(iii)* above, but the noise power is not expected to be great.

The various bounds on the k_{ij} will be such that at every exchange it is the sum of the k_{ij} that must be determined. A device[40] to maintain this sum constant as links are added to an exchange has already been described in Chapter 3 (see fig. 3.2). This will also ensure correct operation in the event of faults.

7.3 Initial connection and prevention of wrong modes

The study of transients shows that the ideal instant for connection of the control system is when all the phase differences are near zero. In practice, the network could be built up progressively, exchanges being added to the control when their phases came close to those of their connected neighbours. This process could be automated so that the network can be restored quickly after faults have been cleared.

As all the initial frequencies are likely to be close to each other, and the system frequency is a weighted mean of the initial frequencies, the time taken for a new exchange to acquire the same phase as the rest of the network is likely to be long. This can be reduced if it is arranged that the new exchange is first 'slaved' to one of the other exchanges. This is possible if at each exchange there is a chain designed to indicate a 'parent' or an alternative if the parent is faulty or no longer connected. Each exchange will in any case send pulses to the new exchange, so that the phase differences can be determined at the new exchange. The various control signals can be added as soon as the phase differences come within some predetermined limit.

The same mechanism, with an added check facility[41], can be used to eliminate wrong mode operation. The most important result that may be derived from the study of wrong mode operation in Chapter 5 is that no wrong mode can occur inside the hypercube of the state space for which $|x_i| < \frac{1}{4}$. This hypercube is always within the region in which no phase comparator has overflowed. The phase differences x_i are taken with respect to a common reference phase.

The design of the check facility depends upon the fact that in a practical network the exchanges are arranged in an heirarchic manner for traffic reasons. Thus exchange is connected to some central exchange via a limited number of intermediate exchanges. The central exchange would be any one of those in the basic trunk telephone network, which is fully interconnected. In a particular example, there would be at most two intermediate exchanges and three links between any one exchange and the central reference exchange.

By means of special signals, it is proposed that every exchange should determine its own phase difference relative to the reference. If this difference should exceed ¼ for more than a short time, it may be infered that the network is in a wrong mode. Such an alarm would be broadcast throughout the network, and cause every exchange to synchronise to its parent alone, as for the initial set-up of the control system. The resultant 'tree' network cannot support a wrong mode, and the correct mode will automatically be selected. As soon as the alarm signal is removed, the control can assume its more fully connected form.

An alternative form of alarm[41] makes use of the limited number of links between the outer exchanges and the central reference point. For a phase difference of more than ¼ frame at least one of the (here) three links must have a phase difference of more than $\frac{1}{12}$ frame across it. This is here two slots, with a 24 channel system. As the allowable phase difference with a one slot capacity buffer store is only ± ½ slot, this may also be interpreted as an alarm if the condition persists. It has the advantage that no special signals are needed to create the alarm signal, but would not be suitable for networks in which large steady-state phase differences were expected.

7.4 Faults and security

The control system must be designed so that the service is uninterrupted by minor disturbances and the effects of faults must be localised within the network.

A complete failure of the exchange clock must be a very rare occurence; an exchange battery supply is expected to give no more that one interruption of under half an hour in 40 years. The same order of reliability is demanded of the clock. For this reason the clock will be compounded of three oscillators with a special locking based upon a majority decision technique. Although such clocks are by design highly stable, it is still possible to control the frequency within limits. The frequency correction system is designed so that it is possible to reject excessive demands for correction, such as would be generated by a faulty control system. However, this would have the effect of isolating the exchange, which is equally undesirable. Thus the design of the common equipment must be protected by a duplicate system. This must include the amplifiers that drive the oscillators together with the mode correction equipment. This is compatible with the system illustrated in fig 3.2.

It is expected that line equipment failures will be more common. One fairly frequent failure is of the line system. If this fails to recognise the incoming synchronising signals, the resultant phase difference signal will be meaningless. On p.c.m. routes with more than one 24 channel system it is advantageous to have the systems divided between different cables, if possible using different routes. A cable fault will still leave some systems working, and if a phase comparator is fitted to at least one of the systems in each cable a continuous phase difference signal is available.

The outputs of different phase comparators between the same two exchanges may be combined by an adding system similar to that used for combining the signals from different exchanges. The sum of the gains used will be unity, so that the output is the same as that which would have been obtained from a single comparator. If a single-ended system is in use, the output will vary according to the mean delay variation over all of the connected p.c.m. systems. This equipment will also need some form of back-up system. The use of several parallel signals enables faulty equipment to be detected and isolated.

One difficulty that arises from the use of a common value for the sum of the gains at each station, with each input to the station having the same gain as every other input, is that the final operating frequency may vary. As the number of working inputs varies, so will the gain applied to any one of them. This the constant k_i used in the formula for the system frequency. This would cause the network frequency to be weighted in favour of the most connected exchange. However, as this is likely to be at a point of some importance in the network, such as an international 'gateway' exchange, this may be of some practical use.

An advantage of synchronised systems is that after a failure of the link synchronisation circuits, the exchange phases are held by the other connections to the rest of the network. When the link circuits are restored there is no need for an extended hunt at the incoming exchange for the phase of the distant exchange, as this is known to within a small number of bits. This can reduce the time for which a link is unusable should it be disturbed by an electrical discharge or other transient phenomenon.

7.5 Growth of the network

In the early stages, a synchronised p.c.m. network is likely to consist of less than ten exchanges. Such a network will approach fully-interconnected status, and may consist of a number of tandem exchanges in an urban area. As the p.c.m. switching technique spreads, more exchanges will be added as described in sections 7.1 and 7.3. This development is likely to occur in parallel in other areas, until eventually it is required that two or more small networks be joined to form a larger network.

To join networks together will require considerably more care than the addition of a new exchange to an existing network. Ideally, two exchanges, one in each area, should be selected which are connected to a large proportion of the exchanges in their areas. These should be joined first, as if a new network consisting of these two exchanges only were being formed. The gains used in the synchronising control between these two exchanges must of necessity be much smaller than the gains used within an area to maintain synchronisation. This will ensure that all the exchanges follow the movement of phase as the two areas are drawn together, without endangering traffic.

An essential preliminary task is to ensure that after the new network has been formed the gains used do not violate the stability criteria. The new values should be introduced before the connection is made. Since the link between the areas is likely to involve longer delays than have been used within the areas, all the gains will have to be reduced by adjustment of amplifiers at affected points. The addition of the remaining links between the two areas may be done automatically as the phase differences approach zero.

It is suggested that minor exchanges should be slaved from a single parent, and have no influence on the control of the rest of the system. If a route to another exchange is available, this can supply the back-up to the parent route. This second route will require a larger buffer store than would be the case if the minor exchange were synchronised to both exchanges. However, a large value of gain may be used, within limits set by noise performance, to make the use of the buffer stores more efficient.

An international digital network could be synchronised over a backbone of satellite and cable connections. However, the practical difficulties may cause the selection of a global network of synchronised satellites and individual national synchronised systems with special interfaces at national boundaries. The proportion of total traffic using these interfaces may lead to this solution on economic grounds.

7.6 A field trial and some theoretical studies

It is now possible to justify the expense of a field trial experiment. In addition, some further measurements need to be made of a p.c.m. system working with live traffic. The two may be combined, if authority can be given to the use of the trial system for live traffic.

An ideal network for a field trial should allow the establishment of a network of at least five exchanges in a fully interconnected network. This will allow the results of section 3.2.3 to be tested, as well as allow measurements of transient response to be made. Additional exchanges that can be connected to some of the basic network would be useful.

One series of tests should cover the stability of both the double-ended and single-ended systems. For this reason the line units used for the trial will need to be more versatile than those used for a production system. In the case of fully-interconnected systems, the symmetry allows the gain margin afforded by the stability criteria in this Thesis to be established. It is thought likely that a considerable gain margin exists for double-ended systems with filters. The presence of noise is likely to prejudice the examination of systems without filters, but this should be attempted for comparison. The behaviour of a fully interconnected system near its stability limit should be investigated as a link is removed. The criteria available for systems less than fully interconnected demand lower gains than those usable for the fully connected systems. The safety margins of these criteria should be tested.

With a set of initial frequencies suitably chosen, it will be possible to examine the effect of gain variations on final frequency; the effects should be as predicted in Chapter 4. Some additional theoretical work may be done in connection with the single-ended systems. Once the actual values of the gains are evaluated, the effect of the expected annual delay variation should be established. If this causes only a small change in network frequency, this may influence the choice of type of system. The results of this study should be confirmed by measurement over a period of time.

It is possible that the effects of noise on system frequency will make the difference between double- and single-ended systems less important. This can be tested by observation over a period of time with and without the injection of noise test signals.

The effects of noise on the buffer store fills needs much examination. An early priority is a series of tests to determine the noise power spectrum. Once this is known, more theoretical work may follow on the expected phase variations due to the noise. If the gains have to be limited, tests of the transient response will show if the buffer stores have sufficient capacity to prevent loss of traffic after surges in the control signals. Some preliminary work[42] has suggested that the noise signals may well have a 'white noise' characteristic below 96 kHz. The amplitude of the signals is, however, unknown.

The transient response of a variety of network configurations should be examined. These should include both highly and weakly connected systems, as defined in section 6.3.2, as well as fully interconnected networks. The simultaneous plotting, at a central point, of transients at all stations will reveal whether the decay times are short enough. Initial start with large phase differences should be tested. The use of Liapunov functionals, to estimate transients for systems with delays, should be examined in detail.

The problem of operation modes can also be investigated. The tests of transient response will reveal whether starting points in the $\underline{\lambda} = \underline{0}$ region do actually cause a wrong mode to be entered. Once a wrong mode has been entered, the detection equipment can be tested. If this is fitted, there is no reason to start a network from any particular initial state, as the selection of the correct mode will be automatic.

The work described in this Thesis attempts to establish the factors governing the behaviour of a synchronisation control system. The field trial will provide an opportunity for the practical engineering of the system to be undertaken and tested. There is every reason to expect the trial to be successful; an economic study must then follow to prove the worth of synchronised networks.

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9 APPENDIX

9.1 Final frequency

The following theorems allow the deduction of the final frequency. Theorem I shows that all the frequencies approach a common value, and Theorem II permits the deduction of eqn. (*4.2*) from eqn. (*4.1*).

Theorem I

The cofactors A_{0qr} of the element a_{0qr} in det.$|\underline{A}(0)|$ are such that for all q, r, s $A_{0qr} = A_{0qs}$.

Proof

$$a_{0ii} = -\sum_{\substack{j=1 \\ \neq i}}^{n} a_{0ij} \quad \text{for all } i, j.$$

Expanding det.$|a_{ij}|$ = det.$|s\delta_{ij} - a_{0ij}|$ where δ_{ij} is, here, the Kronecker delta

$$\text{det.}a_{ij} = -s \begin{array}{c} \left| \begin{array}{cccc|c|cc} a_{11} & a_{12} & \cdot & \cdot & 1 & \cdot & a_{1n} \\ a_{21} & a_{22} & & & 1 & & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & 1 & \cdot & a_{nn} \end{array} \right| \\ \quad r-1 \text{ columns} \qquad r\text{th} \qquad n-r \end{array}$$

by addition of all columns to the rth column.

The coefficient of s^1 is $-\sum_{q=1}^{n} A_{0qr}$ and must be constant for all values of r. Thus $A_{0qr} = A_{0qs}$ for all values of r, s. Q.E.D.

Theorem II

The cofactors A_{0qr} of the element a_{0qr} in det.$|\underline{A}(0)|$ are related by

$k_q A_{0qr}$ = constant, all q, r in the special case

where $a_{0qr} = k_q c_{qr}$ $(q \neq r)$; $a_{0qq} = -k_q \sum_{\substack{r=1 \\ \neq q}}^{n} c_{qr}$

and $c_{qr} = c_{rq} = 1$ when the qth and rth stations are directly connected and $c_{qr} = c_{rq} = 0$ when they are not.

In this case $\underline{A}(0) = \underline{K}.\underline{C}$ where $\underline{K}$ is the diagonal matrix of the k_q and $\underline{C}$ is the matrix of the c_{qr}.

Proof

From the above conditions it follows that $\sum_{r=1}^{n} a_{0qr} = 0$

and $\sum_{q=1}^{n} (a_{0qr}/k_q) = 0$

Consider now the expansion of det.$|a_{ij}|$ = det.$|s\delta_{ij} - a_{0ij}|$ where δ_{ij} is the Kronecker delta.

Adding all columns to rth column,

$$\det.|a_{ij}| = -s \begin{array}{c} \begin{array}{ccc} r-1 & r\text{th} & n-r \end{array} \\ \left|\begin{array}{c|c|c} & 1 & \\ & 1 & \\ & 1 & \\ & 1 & \\ & 1 & \\ & 1 & \end{array}\right| \end{array}$$

$$= -s \prod_{i=1}^{n} k_i \left|\begin{array}{c|c|c} \dfrac{a_{ij}}{k_i} & \dfrac{1}{k_i} & \dfrac{a_{ij}}{k_i} \end{array}\right|$$

dividing ith row by k_i for all i

$$= -s \prod_{i=1}^{n} k_i \left|\begin{array}{c|c|c} \dfrac{a_{ij}}{k_i} & \dfrac{1}{k_i} & \dfrac{a_{ij}}{k_i} \\ \hline \dfrac{s}{k_j} & \Sigma \dfrac{1}{k_i} & \dfrac{s}{k_j} \\ \hline \dfrac{a_{ij}}{k_i} & \dfrac{1}{k_i} & \dfrac{a_{ij}}{k_i} \end{array}\right| \begin{array}{l} q-1 \\ \\ q\text{th row} \\ \\ n-q \end{array}$$

adding all rows to qth row

$$= -s \left|\begin{array}{c|c|c} a_{ij} & 1 & a_{ij} \\ \hline s\,\dfrac{k_q}{k_j} & k_q \Sigma \dfrac{1}{k_i} & s\,\dfrac{k_q}{k_j} \\ \hline a_{ij} & 1 & a_{ij} \end{array}\right|$$

multiplying ith row by k_i for all i

This expansion may be regarded as a power series in s. The coefficient of s^1 is given by

$$- k_q \sum_{i=1}^{n} (1/k_i) A_{0qr} \text{ for all } q, r.$$

Since this does not depend on the method of evaluation, $k_q A_{0qr}$ is constant for all q, r. Q.E.D.

9.2 Operating mode

An alternative to the use of the matrix $\underline{A}^*$ in section 5.1 is the use of the matrix $\underline{A}^o$ defined by the deletion of the nth row and column. The following theorem shows that the modes thus obtained are identical.

Theorem III

$\underline{A}^*$ and $\underline{A}^o$ are such that for a given vector $\underline{x}$ the vectors $\underline{\lambda} = \underline{A}^*.\underline{x}$ and $\underline{\lambda}^o = \underline{A}^o.\underline{x}$ are identical.

Proof

$$\underline{A}^* = \begin{bmatrix} a_{11} - a_{n1} & a_{12} - a_{n2} & \cdot & \cdot & a_{1,n-1} - a_{n,n-1} \\ a_{21} - a_{n1} & a_{22} - a_{n2} & \cdot & \cdot & a_{2,n-1} - a_{n,n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1,1} - a_{n1} & a_{n-1,2} - a_{n2} & \cdot & \cdot & a_{n-1,n-1} - a_{n,n-1} \end{bmatrix}$$

$= \underline{A}^o - \underline{H}$ where

$$\underline{A}^o = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1,n-1} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2,n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1,1} & a_{n-1,2} & \cdot & \cdot & a_{n-1,n-1} \end{bmatrix}$$

and $\underline{H} = \left[a_{nj}\right]$ for $j = 1, 2, \ldots, n - 1$.

Now in the steady state, the phase differences are such that $\underline{x}$ is defined by

$$\underline{\lambda} = \underline{A}^*.\underline{x} \text{ or } \underline{\lambda}^o = \underline{A}^o.\underline{x} = \underline{A}^*.\underline{x} + \underline{H}.\underline{x}$$

$$= \underline{\lambda} + \underline{H}.\underline{x}$$

In the basic hypercube $q_{ni} = 0$ and $x_i = \phi_i - \phi_n$. In the steady state,

$$\dot{\phi}_n = 0 = \sum_{j=1}^{n} a_{nj}(\phi_j - \phi_n) = \sum_{j=1}^{n} a_{nj}.x_j \quad \text{from eqn. } (5.1.7)$$

Thus $\underline{H}.\underline{x} = \underline{0}$ whence $\underline{\lambda} = \underline{\lambda}^{\circ}$ Q.E.D.

It is also possible to derive an additional property of the connection matrix from the above theorems.

Theorem IV

Where $a_{ij} = a_{ji} = 1$ for the ith and jth exchanges to be connected and zero otherwise, the matrix $\underline{A} = |a_{ij}|$ has its adjugate such that

$$\text{adj. } \underline{A} = \frac{1}{n} \text{ det. } |\underline{A}^*| \cdot \begin{bmatrix} 1 & 1 & \cdot & 1 \\ 1 & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 \end{bmatrix} (n-1) \times (n-1)$$

Proof

Consider

$$\text{det. } |s\underline{I} - \underline{A}| = \text{det. } |s\delta_{ij} - a_{ij}|$$

$$= \left| \begin{array}{c|c} s\delta_{ij} - a_{ij} + a_{nj} & -a_{in} - s - m_n \\ & \\ \hline -a_{nj} & s + m_n \end{array} \right| \begin{array}{l} n-1 \text{ rows} \\ \\ n\text{th row} \end{array}$$

$$\begin{array}{cc} n-1 \text{ columns} & n\text{th column} \end{array}$$

by subtraction of the nth row from all the other rows. Note that

$$m_i = \sum_{\substack{j=1 \\ \neq i}}^{n} a_{ij}$$

Adding all columns to the nth column

$$\det. |s\underline{I} - \underline{A}| = \left| \begin{array}{c|c} s\delta_{ij} - (a_{ij} - a_{nj}) & \underline{0} \\ \hline -a_{nj} & s \end{array} \right|$$

$$= s.\det. |-\underline{A}^*| + \text{higher powers of s}$$

Applying Theorem II and comparing coefficients,

$$\det. |\underline{A}^*| = n\, A_{oqr}$$

where A_{oqr} is the cofactor of a_{qr}.

The cofactors A_{oqr}, which are all equal, are the r, qth elements of the adjugate of $\underline{A}$. Thus

$$\text{adj}.\ \underline{A} = \frac{1}{n} \det. |\underline{A}^*| \cdot \left| \begin{array}{ccc} 1 & 1 \cdot & 1 \\ 1 & 1 \cdot & 1 \\ \cdot & \cdot \quad \cdot & \cdot \\ 1 & 1 \cdot & 1 \end{array} \right| \qquad \text{Q.E.D.}$$

In the case of a fully interconnected network $\det. |\underline{A}^*| = n^{n-1}$.

Corollary

The rank of matrix $\underline{A}$ is $n - 1$.

Since adj. $\underline{A} \neq \underline{0}$, rank $\underline{A} \geq n - 1$. But det. $|\underline{A}| = 0$, so rank $\underline{A} < n$. Thus rank $\underline{A} = n - 1$.

9.3 Published papers

Part of the work submitted in this Thesis has already been published. Copies of the relevant papers follow. In order of publication these are

December	1966	PARKS, P. C. and MILLER, M. R. 'Stability of p.c.m. characteristic equations' Electron. Lett., 1966, *2*, pp. 468-469
May	1967	PARKS, P. C. and MILLER, M. R. 'Liapunov estimates of oscillator transients in p.c.m. systems' *ibid.*, 1967, *3*, pp. 193-194
January	1968	PARKS, P. C. and MILLER, M. R. 'Stability of p.c.m. oscillator control systems' *ibid.*, 1968, *4*, p. 26
June	'1968	MILLER, M. R. and PARKS, P. C. 'Phase plane trajectories for linear p.c.m. oscillator control systems' *ibid.*, 1968, *4*, pp. 171-173
May	1969	MILLER, M. R. 'Stability of p.c.m. oscillator synchronisation systems with sampled error signals' *ibid.*, 1969, *5*, pp. 183-184
July	1969	MILLER, M. R. 'Feasibility studies of synchronised oscillator systems for p.c.m. telephone networks' Proc. IEE, 1969, *116*, (7), pp. 1135-1143
July	1969	PARKS, P. C. and MILLER, M. R. 'Results of some studies of p.c.m. network synchronisation systems' Automatica, 1969, *5*, (3), pp. 449-463

These are cited as references 11, 17, 21, 27, 35, 36, 37 respectively. The last is a revised version of the paper given at the IFAC Symposium on Multivariable Systems at Düsseldorf in October 1968 (see ref. 29).

STABILITY OF P.C.M. CHARACTERISTIC EQUATIONS

The stability criteria of an oscillator control system suitable for a network of p.c.m. systems are examined. The system is shown to be stable for a wide range of signal-path delay.

In stability studies of interconnected oscillator systems for p.c.m. communication, we are led to investigate the following $n \times n$ determinantal characteristic equation in p:

$$\det \{a_{ij} \exp(-pd_{ij}) - \delta_{ij} p\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where δ_{ij} is the Kronecker delta (i.e. $\delta_{ii} = 1$; $\delta_{ij} = 0$, $i \neq j$), and we distinguish the following six special cases:

(i) zero system delays (all d_{ij} zero), $a_{ij} = 0$ or $k (i \neq j)$ and

$$a_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} (a_{ij})$$

$[a_{ij}]$ is the system 'connection matrix' and is usually symmetric

(ii) zero system delays, $a_{ii} = -k$, $a_{ij} = 0$ or $\dfrac{k}{m_i} (i \neq j)$;

so that $a_{ii} = -k = \sum_{\substack{j=1 \\ j \neq i}}^{n} (a_{ij})$

(iii) a_{ij} as in (i), but d_{ij} nonzero for $i \neq j$, $d_{ii} = 0$
(iv) a_{ij} as in (ii), d_{ij} as in (iii)
(v) a_{ij} as in (i), all $d_{ij} = d$, where $d = \max(d_{ij})$ in (iii)
(vi) a_{ij} as in (ii), $d_{ij} = d$ as in (v)

In these six cases, a_{ij} are all real, d_{ij} and k are real and positive, and m_i is a positive integer by definition.

The stability of cases (i) and (ii) may be established at once by use of the Gershgorin circle theorem,[1] which states that the eigenvalues of the $n \times n$ matrix $[b_{ij}]$ lie within the union of n circles in the complex plane, with centres b_{ii} and radii $\sum_{\substack{j=1 \\ \neq i}}^{n} |b_{ij}|$. Here, the ith circle has its centre at the point $(a_{ii}, 0)$

in the negative half of the complex p plane and radius $-a_{ii}$. There is a zero root in p, since $\det[a_{ij}] = 0$; but there are certainly no roots in p with positive real parts. In case (i), the nonzero roots will be real and negative if $[a_{ij}]$ is symmetric and the system will be aperiodic. In case (ii), the matrix $[a_{ij}]$ is not symmetric; so that complex conjugate pairs of roots are possible, all the roots being confined, however, to the circle centre $(-k, 0)$ with radius k. There is again a zero root.

Cases (iii) and (iv) could be treated by assuming that the delays are negligible, giving cases (i) and (ii), respectively. However, we wish to consider a 'worst case' by extending all the delays to equal the longest delay in the system, thus obtaining cases (v) and (vi). The characteristic roots in p are now the solutions of the equations

$$p \exp(pd) = \lambda_i \quad (i = 1, 2 \ldots n) \qquad (2)$$

where the λ_i are the roots in p of cases (i) and (ii), respectively.

Considering case (v) with a symmetric $[a_{ij}]$, we know that the λ_i are real negative numbers in the range $-2k(n-1), 0$. For a given nonzero λ_i, eqn. 2 is the characteristic equation of a simple feedback control system consisting of an integrator with transfer function $-\lambda_i/p$ in the forward path and a pure time delay d in the feedback path. Applying the Nyquist stability criterion to such a system,[2] we obtain the condition that $(-\lambda_i)d < \frac{\pi}{2}$. Considering the largest possible $(-\lambda_i)$, we obtain a sufficient condition for stability that

$$(n-1)kd < \frac{\pi}{4} \qquad (3)$$

In case (vi), similar considerations apply with the complication of complex roots λ_i arising from case (ii). However, by noting that the $-\lambda_i$ lie inside a circle centre $(+k, 0)$ and radius k, and by applying the Nyquist criterion to the control system above, with a complex gain in the integrator, we obtain a corresponding sufficient condition that

$$kd < \tfrac{1}{2} \qquad (4)$$

If the network configuration were such that the λ_i in case (ii) were known to be real, expr. 4 could be relaxed to

$$kd < \frac{\pi}{4} \qquad (5)$$

by the method of case (v).

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LYAPUNOV ESTIMATES OF OSCILLATOR TRANSIENTS IN P.C.M. SYSTEMS

Lyapunov techniques are applied to the problem of estimating phase overshoots in a p.c.m. system with controlled oscillators. Two possible approaches are examined for use in highly and sparsely connected networks, respectively. The case of full interconnection is also dealt with.

In a study of the synchronisation of a network of controlled oscillators, such as a p.c.m. system, we need to investigate the phase errors between stations during transient conditions. These determine the size of the buffer stores between the stations.

In a previous letter,[1] the stability of various systems was discussed. The present study concerns case (i) of this previous letter.

The system equations are:

$$\dot{f} = Af \quad \ldots \quad (1)$$

where f is the vector of the station frequencies, and A is the system matrix defined by

$$a_{ij} = 0 \quad \text{or} \quad k(i \neq j) \quad \text{and} \quad a_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} (a_{ij})$$

The matrix is symmetric. When the network is fully interconnected, $a_{ii} = -(n-1)k$ and $a_{ij} = k(i \neq j)$.

Subtracting components in eqns. 1, we obtain

$$\dot{f}_i - \dot{f}_j = -nk(f_i - f_j) \quad \ldots \quad (2)$$

Integrating, we obtain

$$(f_i - f_j)_t = (f_i - f_j)_0 \, e^{-nkt} \quad \ldots \quad (3)$$

and

$$(\phi_i - \phi_j)_t = (\phi_i - \phi_j)_0 + \int_0^t (f_i - f_j)_0 \, e^{-nkt} dt$$

$$= (\phi_i - \phi_j)_0 + (f_i - f_j)_0 \left(\frac{1 - e^{-nkt}}{nk} \right) \quad (4)$$

In general, the phase difference will also be the initial phase difference plus the integral of the frequency difference between the stations, and, in this letter, we show how some bounds for this integral may be deduced with the minimum knowledge of the form of the system matrix A.

We know from stability studies that all the $f_i (i = 1, 2, \ldots n)$ tend to the same value f_∞ as $t \to \infty$, and, by adding the n equations, that $\sum_{i=1}^{n} (\dot{f}_i) = 0$, whence $\sum_{i=1}^{n} (f_i) = \text{const.} = nf_\infty$.

To consider transient motion, we introduce a new variable defined by

$$y_i = f_i - f_\infty \quad \ldots \quad (5)$$

Hence

$$\sum_{i=1}^{n} (y_i) = 0 \quad \ldots \quad (6)$$

We shall estimate the integral of frequency difference by using the following equalities and inequalities:

$$\left| \int_0^t (f_i - f_j) dt \right| = \left| \int_0^t (y_i - y_j) dt \right| \leqslant \int_0^t |y_i - y_j| dt$$

$$\leqslant \int_0^\infty |y_i - y_j| dt \leqslant \int_0^\infty 2\rho(t) dt$$

$$\leqslant \int_0^\infty 2R \, e^{-qt} dt = \frac{2R}{q}$$

where $\rho(t)$ is the Euclidean state–space norm, defined by

$$\rho^2(t) = y'(t)y(t) \quad \ldots \quad (7)$$

$R e^{-\gamma t}$

An upper bound on $\rho(t)$ is obtained by use of the Lyapunov function method of estimating transients.[2,3]

We use the Lyapunov function $V = y'y$, for which $dV/dt = 2y'Ay$. The problem now is to find an inequality of the form

$$\frac{dV}{dt} \leqslant -cy'y$$

from which we may conclude that

$$\rho(t) \leqslant \rho(0) \exp(-\tfrac{1}{2}ct)$$

where $\rho(0)$ is determined by the initial frequencies.

However, A as it stands is negative semidefinite, but the y_i are constrained to satisfy the condition

$$y_1 + y_2 + y_3 \ldots + y_n = 0$$

We may therefore add to dV/dt the additional quadratic form

$$-2\mu^2 k(y_1 + y_2 + y_3 \ldots + y_n)^2$$

Thus

$$\frac{dV}{dt} = -2\{y'(-A)y + \mu^2 k y' B y\}$$

where $B_{ij} = 1$ (all i, j).

We now distinguish two cases:

(a) highly connected networks, in which each station i is connected to at least m_i other stations, where $m_i \geqslant n/2 - 1$

(b) sparsely connected networks where (a) is not satisfied.

In case (a), an immediate estimate $c = 2k[2(m_i + 1) - n]_{min\,i}$ ($i = 1, 2, \ldots n$) is available from the Gershgorin circle theorem[4] applied to the matrix

$$[-A + \mu^2 kB] \quad \text{with } \mu = 1$$

giving

$$R = \rho(0), \quad q = k2[2(m_i + 1) - n]_{min\,i} \quad (i - 1, 2, 3 \ldots n) \quad \ldots \quad (8)$$

Case (b) may be treated by first considering networks in which a chain connecting stations $1, 2, 3 \ldots n$ in successive order exists for a suitable ordering.

For networks with such a chain,

$$\frac{dV}{dt} \leqslant -2k \left\{ \sum_{i=1}^{n-1} (y_i - y_{i-1})^2 \right\}$$

and as before we may add on $-2\mu^2 k(y_1 + y_2 + y_3 \ldots + y_n)^2$ to yield

$$\frac{dV}{dt} \leqslant -2ky'T'Ty$$

where

$$T = \begin{pmatrix} 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & -1 & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & -1 \\ \mu & \mu & \mu & . & \mu & \mu \end{pmatrix} \quad (n \times n)$$

To obtain a new value of c, we need to know the smallest latent root of $T'T$. Now T is nonsingular for nonzero μ, and the latent roots of $T'T$ are equal to the latent roots of TT'.

$$\text{Now } TT' = \begin{pmatrix} 2 & -1 & & 0 & \\ -1 & 2 & & & \\ & & & -1 & 0 \\ & & -1 & 2 & 0 \\ 0 & & 0 & 0 & n\mu^2 \end{pmatrix} \quad (n \times n)$$

of which the latent roots are $n\mu^2$ and $4 \sin^2 (r\pi/2n)$ ($r = 1, 2, 3 \ldots n - 1$). We choose μ so that the

$$n\mu^2 > 4 \sin^2 \frac{\pi}{2n}$$

and thus obtain

$$c = 8k \sin^2 \frac{\pi}{2n} \quad \ldots \quad (9)$$

Again $R = \rho(0)$ and $q = c/2$.

Spurs can be dealt with by counting stations twice to form a chain of length greater than n stations, and using the method described above. The estimate will be increased by:

(a) the effect on both R and c of the additional stations

(b) the effect on R of the largest number of times a given frequency is used as a state variable.

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STABILITY OF P.C.M.-OSCILLATOR CONTROL SYSTEMS

An extension of existing techniques allows a proof of the stability of two classes of linear oscillator control system for use in integrated p.c.m. systems. Stability is shown to depend on the response of various filters used in the system in a manner which should be easily realisable in a practical network.

In a previous letter,[1] the stability of various types of oscillator control system was discussed. Two cases, (*iii*) and (*iv*) in that letter, were left unsolved. Recently West[2] has indicated a proof of the stability of the former special case (*iii*). However, we are now able to extend this method to examine the stability of the general case which includes both of the unsolved special cases. We are also able to examine the stability of another class of system[3] in (*b*) below.

The system equations are of the form

$$\{sI - A(s)\}f = c$$

where the vector c determines the final system frequency and is a function of the initial conditions. The characteristic equation determining stability is

$$\det | sI - A(s) | = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the roots in $s = \sigma + j\omega$ of eqn. 1 have positive real parts ($\sigma > 0$), the system is unstable. Thus the matrix $B(s) = | sI - A(s) |$ may not be singular for $\sigma \geqslant 0$, $j\omega = 0$, if the system is to be stable. If for some $s = s_1$ this is singular, $\lambda = 0$ is a root of

$$\det | \lambda I - B(s_1) | = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We now use the Gershgorin circle theorem[4,5] to locate the eigenvalues λ of the matrix $B(s_1)$ from eqn. 2. This states that the eigenvalues lie in that region of the complex plane which is the union of the circles whose centres are $b_{ii}(s_1)$ and whose radii are $\sum_{\substack{j=1 \\ \neq i}}^{n} | b_{ij}(s_1)|$ for $i = 1, 2, \ldots, n$ and where $b_{ij}(s_1)$ is the i, j element of $B(s_1)$. These circles cannot contain the origin, whence $\lambda \neq 0$, if for all i

$$|b_{ii}(s_1)| > \sum_{\substack{j=1 \\ \neq i}}^{n} |b_{ij}(s_1)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If expr. 3 is satisfied for s_1 anywhere in the right-hand half of the s plane, the system with characteristic eqn. 1 is stable.

We examine two cases:

(*a*) $b_{ii}(s) = s + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}$ $\qquad b_{ij}(s) = - k_{ij} \exp(- sd_{ij})$

(*b*) $b_{ii}(s) = s + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\{1 + \exp(-sd_{ij} - sd_{ji})\}$

$\qquad b_{ij}(s) = - 2k_{ij} \exp(- sd_{ij})$

where $k_{ij}(> 0)$ is the gain element in the control-signal path at exchange i from exchange j, and d_{ij} is the corresponding line delay.

We now examine the possibility that $B(s)$ is singular for $s = \sigma + j\omega$ in the right-hand halfplane. For $\sigma \geqslant 0$, $| \exp(- sd_{ij}) | \leqslant 1$, and so

$$\sum_{\substack{j=1 \\ \neq i}}^{n} |k_{ij} \exp(- sd_{ij})| = \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} | \exp(- sd_{ij})| \leqslant \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In case (*a*) the system is stable, from expr. 3, provided that for all $\sigma \geqslant 0, j\omega \neq 0$:

$$|s + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}| > \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij} \geqslant \sum_{\substack{j=1 \\ \neq i}}^{n} |k_{ij} \exp(- sd_{ij})| \quad . \quad (5)$$

This is always satisfied. There is a root at $s = 0$, corresponding to the lack of a reference frequency. This result extends to the case of general values of k_{ij} the result first deduced by West, which is that this system is stable for any network configuration in the absence of filters.

For case (*b*), the system is stable in the absence of filters when the gains and delays satisfy the sufficient condition that, for each i,

$$(d_{ij} + d_{ji})_{max\, j,\, j=1,2,\cdots n} \left(\sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\right) < 0{\cdot}415 \quad . \quad (6)$$

This result makes use of expr. 4 above, and the resultant expr. 6 is likely to be satisfied by practical values of the system parameters. It is an improvement on criteria derived for this system by the methods of cases (*v*) and (*vi*) of our previous letter.

Various filters appear in the system design before the gain element and also after the delay element. These will determine the stability conditions of the system, and a sufficient condition on the transfer function $F_{ij}(s)$ of each filter is that, for stability, $|F_{ij}(s)| \leqslant 1$ for all $s \neq 0$ and in the right-hand halfplane. A more stringent condition is $|F_{ij}(j\omega)| \leqslant 1$ for all ω. Sharper criteria may be deduced for particular forms of $F_{ij}(s)$.

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PHASE-PLANE TRAJECTORIES FOR LINEAR-P.C.M.-OSCILLATOR CONTROL SYSTEMS

The existence of more than one operating mode for these systems is well known. Phase-plane diagrams are shown to be useful in deducing the existence and stability of these modes as the line delays and initial uncontrolled oscillator frequencies are varied. Simple 3-station networks are considered here, but the technique may be generalised to larger networks.

The stability of the control system has been deduced[1] by means of the Laplace transform of the system equations. The final-system frequency f_s can also be found from these transformed equations. Phase-plane diagrams may be constructed for the system. The equilibrium points and trajectories in these diagrams give much insight into the system behaviour, including the existence and stability of operating modes. This technique is a powerful alternative to the analysis of Saito,[2] and can be generalised to include large systems.

The system equations are

$$\dot{\phi}_i = f_{0i} + \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}(\phi_{ij} - r_{ij}) \qquad (1)$$

where

$$r_{ij} = [\phi_{ij} + \tfrac{1}{2}] \qquad [\,]\text{ is the Gaussian notation,}$$

and

$$\phi_{ij} = \phi_j - \phi_i - \int_{t-d_{ij}}^{t} \dot{\phi}_j d\tau$$

The term r_{ij} arises from the cyclic nature of the phase comparators.

We will put $k_{ij} = ka_{ij}$, where $a_{ij} = 0$ or 1 and

$$a_{ii} = -\sum_{\substack{j=1\\ \neq i}}^{n} a_{ij}$$

and we also put $d_{ii} = 0$ and $r_{ii} = 0$.

We define δ_{ij} and N_{ij} by $f_s d_{ij} = N_{ij} + f_s\delta_{ij}$ . . (2)

where N_{ij} is an integer and f_s is the final-system frequency.

We define the integer

$$q_{ij} = r_{ij} + N_{ij} \qquad (3)$$

and we find that $q_{ij} = 0$ when $\phi_i = \phi_j$ and $\delta_{ij} = 0$.

We now transform eqns. 1 to phase-difference equations by putting $\boldsymbol{x} = \boldsymbol{B\phi}$, where $\boldsymbol{B}$ is typically of the form

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & . & 0 & -1 \\ 0 & 1 & . & 0 & -1 \\ 0 & 0 & . & 1 & -1 \end{bmatrix}$$

and we define $\boldsymbol{A^*}$ and $\boldsymbol{D^*}$ by $\boldsymbol{A^*.B = B.A}$ and $\boldsymbol{D^*.B = B.D}$.

Eqn. 1 becomes

$$[\boldsymbol{I} + k\boldsymbol{D^*}]\dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{f}_0 + k\boldsymbol{A^*x} - k\boldsymbol{B}[\boldsymbol{R} + f_s\boldsymbol{D}]\boldsymbol{c}$$

where $\boldsymbol{c}' = [1, 1, \ldots 1]$

From eqns. 2 and 3,

$$[\boldsymbol{R} + f_s\boldsymbol{D}] = [\boldsymbol{Q} + f_s\boldsymbol{\Delta}]$$

Hence

$$[\boldsymbol{I} + k\boldsymbol{D^*}]\dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{f}_0 + k\boldsymbol{A^*x} - k\boldsymbol{B}[\boldsymbol{Q} + f_s\boldsymbol{\Delta}] \qquad (4)$$

$$= \boldsymbol{0} \text{ in the steady state for which } \dot{\phi}_i = f_s,$$

for all $i = 1, 2, \ldots, n$.

The matrix $\boldsymbol{\Delta}$ has as elements the variations δ_{ij} of the delays from the ideal values of integral multiples of frames.

The matrix $\boldsymbol{Q}$ has as elements the number of times q_{ij} that the output of the phase comparator has passed through the discontinuity owing to variations of phase, line delay, or oscillator natural frequency, after the initial set-up.

In a fully interconnected network, $\boldsymbol{A^*} = n\boldsymbol{I}$. If, in addition, all the delays are equal to d, $\boldsymbol{D^*} = -d\boldsymbol{I}$ and $\boldsymbol{B}[\Delta \boldsymbol{c}] = \boldsymbol{B}[(n-1)\delta\boldsymbol{c}] = \boldsymbol{0}$.

We will first consider the fully connected network with zero delays; whence Q is skew-symmetric. Eqn. 4 becomes

$$\begin{aligned}\dot{x} &= nkx + Bf_0 - kBQc \\ &= nkx + Bf_0 - k\boldsymbol{\lambda}\end{aligned}$$

where we define $\boldsymbol{\lambda} = BQc$. In the steady state, $\dot{x} = 0$, and so

$$x = \frac{-1}{nk}Bf_0 + \frac{1}{n}\boldsymbol{\lambda} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

The elements of $\boldsymbol{\lambda}$ have the following properties:

(a) $\lambda_i = \sum_{j=1}^{n} q_{ij} - \sum_{j=1}^{n} q_{nj}$

(b) $\sum_{i=1}^{n-1} \lambda_i = n\sum_{i=1}^{n-1} q_{in}$

Eqn. 5 gives the final value of the phase differences, $\boldsymbol{\lambda}$ defining the operating mode. An increase of any x_i in steps of 1 causes the corresponding q_{ni} to increase in steps of 1; so we may restrict our examination to the state-space region $|x_i| < \frac{1}{2}$, all $i = 1, 2, \ldots, (n-1)$, because all other similar regions may be shown to be identical by translation of the origin.

The properties of $\boldsymbol{\lambda}$ for this region of state-space, $|x_i| < \frac{1}{2}$, are

(i) $\lambda_i = \sum_{j=1}^{n-1} q_{ij}$

(ii) $\sum_{i=1}^{n-1} \lambda_i = 0$

The q_{ij} will be zero unless $|x_i| > \frac{1}{4}$, or $|x_j| > \frac{1}{4}$, or both. Thus in the region $|x_i| < \frac{1}{4}$, for $i = 1, 2, \ldots, (n-1)$, only one mode $\boldsymbol{\lambda} = 0$ can exist for this type of network.

For this fully interconnected network,

$$\begin{aligned}q_{ij} &= [x_j - x_i + \tfrac{1}{2}] \\ &= \left[\frac{1}{nk}(f_{0i} - f_{0j}) + \frac{1}{n}(\lambda_i - \lambda_j) + \tfrac{1}{2}\right]\end{aligned}$$

whence

$$\lambda_i = \sum_{j=1}^{n-1}\left[\frac{1}{nk}(f_{0i} - f_{0j}) + \frac{1}{n}(\lambda_i - \lambda_j) + \tfrac{1}{2}\right] \quad . \quad (6)$$

Only vectors $\boldsymbol{\lambda}$ which satisfy eqn. 6 can generate stable operating modes. For $n = 3$, these are $\boldsymbol{\lambda} = \begin{bmatrix}-1\\+1\end{bmatrix}$ or $\begin{bmatrix}1\\-1\end{bmatrix}$. Fig. 1 shows this case, $\boldsymbol{\lambda}$ being constant in each of the regions as shown.

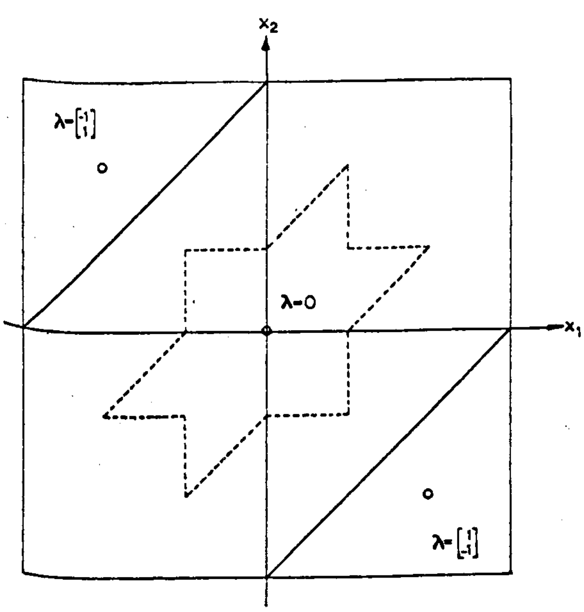

Fig. 1 *Phase-plane for 3-exchange network with zero delays*
O stable mode

If the vector $\frac{1}{nk}Bf_0$ causes the operating point to leave the region of constant $\boldsymbol{\lambda}$, then that mode ceases to exist. In Fig. 1, if this vector, relative to $x = 0$, lies outside the dotted line and inside the region for which $\boldsymbol{\lambda} = 0$, only this mode is stable. For vectors outside this larger area, no operating mode is stable and f_s does not exist. This gives a minimum value for k, the system gain.

When the value of x for $x = 0$ lies on a boundary between two $\boldsymbol{\lambda}$ regions, trajectories of x converge on this point from one region and pass along a common trajectory into the adjacent region, which in this case contains the stable mode $\boldsymbol{\lambda} = 0$.

For nonzero δ_{ij}, $q_{ij} \neq -q_{ji}$ for all $(\phi_i - \phi_j)$. Thus the boundaries between regions of constant $\boldsymbol{\lambda}$ split, and other values of $\boldsymbol{\lambda}$ are generated. These will depend on the sign of δ_{ij} and will not usually satisfy eqn. 6, since the practical range of $f_s\delta_{ij}$ will be too small for the new modes to be engulfed by the region generating them.

In a region for which $q_{ij} \neq -q_{ji}$, trajectories will be straight lines towards the mode point corresponding to the $\boldsymbol{\lambda}$ for that region. Such regions are zones A and B in Fig. 2,

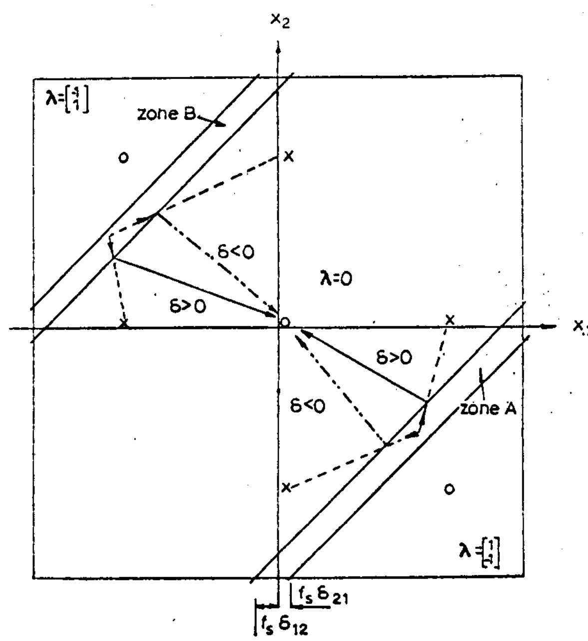

Fig. 2 *Effect of adding small delays while $Bf_0 = 0$*
O stable mode
× unstable mode

and, for $\delta_{12} = \delta_{21} = \delta > 0$, the corresponding values of $\boldsymbol{\lambda}$ are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\end{bmatrix}$ for A and B, respectively. When $\delta < 0$, these become $\begin{bmatrix}0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, respectively. The displacement of the boundaries in both x_1 and $(-x_2)$ directions will be $-f_s\delta_{12}$ and $+f_s\delta_{21}$, so that, when $\delta_{12} + \delta_{21} = 0$, these zones are of zero width. The mode points are also displaced by $(1/3)f_s\begin{pmatrix}\delta_{12}\\\delta_{21}\end{pmatrix}$.

As soon as a boundary of these regions is encountered, the trajectory will change direction, and a stable operating mode will be reached as shown in Fig. 2.

As $B[f_s\Delta]c$ and Bf_0 are varied, one of the following sets of points describes the possible stable modes:

Correct mode

$$\boldsymbol{\lambda} = \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

Wrong mode

$$\boldsymbol{\lambda} = \begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

Wrong mode

$$\boldsymbol{\lambda} = \begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

The first set corresponds to $(f_sB\Delta c) = 0$ and $Bf_0 = 0$.

Higher-order systems will generate higher-order phase spaces, but the general rule holds that the only stable operation modes are those which lie in the λ region generating them.

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STABILITY OF P.C.M.-OSCILLATOR SYNCHRONISATION SYSTEMS WITH SAMPLED ERROR SIGNALS

Previous analyses of control-system stability have approximated the p.c.m. synchronisation system to a continuous linear model. The digital nature of the equipment indicates that a sampled model is more appropriate, and it is shown that, with this model, the system gains have to be reduced. A comparison between 'continuous' and 'sampled' stability criteria is made.

We will consider the operation of a double-ended bilateral control system,[1] as applied to the problem of p.c.m.-oscillator synchronisation. The single-ended system is a special case of this.

The sampling is performed with period T_s seconds, and pulse duration w seconds. We will use the z transformation to analyse the system, putting $z = e^{sT_s}$.

The line from exchange j to exchange i has delay d_{ij} seconds, and the z transfer function, because of the digital nature of the controls, is $z^{-D_{ij}}$, where $D_{ij} = [1 + d_{ij}/T_s]$; we use the Gaussian-bracket notation that $[x]$ is the integer, so that $[x] \leqslant x < [x+1]$. At exchange i, the phase of its own oscillator is subtracted from the apparent phase of the oscillator of exchange j, as received over the line between the exchanges. This phase difference is sampled and passed into a digital filter with transfer function $F^*_{ij}(z)$. The corresponding phase difference formed at exchange j is passed over a data link, with transfer function $z^{-\delta_{ij}}$, where $\delta_{ij} = [1 + \tau_{ij}/T_s]$ and τ_{ij} seconds is the data-link time delay to exchange i. There it is passed through a digital filter with transfer function $H^*_{ij}(z)$ and subtracted from the filtered difference produced at i. The resultant error signal is given a weighting, or gain, k_{ij} before being added to all the other error signals available at exchange i. The combined signal is then used, via a zero-order hold circuit, to modify the frequency $f_i(t)$ of the oscillator at exchange i.

We obtain the system equations

$$\left[1 + \frac{T_s}{z-1} \sum_{\substack{j=1 \\ j \neq 1}}^{n} k_{ij} \{F^*_{ij}(z) + H^*_{ij}(z) z^{-(\delta_{ij} + D_{ji})}\}\right] f^*_i(z)$$

$$- \frac{T_s}{z-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} \{F^*_{ij}(z) z^{-D_{ij}} + H^*_{ij}(z) z^{-\delta_{ji}}\} f^*_j(z) = u^*_i(z)$$

$$i = 1, 2, \ldots, n \qquad \ldots \ldots \ldots \qquad (1)$$

where $f^*_i(z)$ is the sampled form of $f_i(t)$, and the variable $u^*_i(z)$ is determined by the initial conditions. These equations can be written in the matrix form

$$A^*(z) f^*(z) = u^*(z) \qquad \ldots \ldots \ldots \qquad (2)$$

For stability, the matrix $A^*(z)$ must not be singular for $|z| \geqslant 1$, $z \neq 1$. The root at $z = +1$ corresponds to the lack of an external reference frequency, as with the 'continuous' system model.

We use the 'diagonal dominance' version of the Gershgorin circle theorem,[2] as for the continuous systems,[1,3] to obtain the sufficient criterion that, for each $i = 1, 2, 3, \ldots, n$,

$$|a^*_{ii}(z)| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a^*_{ij}(z)| \text{ for all } |z| \geqslant 1 \qquad \ldots \qquad (3)$$

We consider the following cases:

(*a*) *Single-ended system* $[H^*_{ij}(z) = 0]$ *with delays* $(D_{ij} \neq 0)$ *but no filters* $[F^*_{ij}(z) = I]$:

Expr. 3 becomes:

$$\left| z - 1 + T_s \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} \right| > T_s \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} \geqslant T_s \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} |z^{-D_{ij}}|$$

for all $|z| \geqslant 1$. We see that the required condition is that, for all i,

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} < \frac{1}{T_s} \qquad \ldots \ldots \ldots \qquad (4)$$

(*b*) *Single-ended, but with filters* $[F^*_{ij}(z) \neq I]$:

We require that, for all $|z| \geqslant 1$,

$$\left| z - 1 + T_s \sum_{\substack{j=1 \\ j \neq i}}^{n} \{k_{ij} F^*_{ij}(z)\} \right| > T_s \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} |F^*_{ij}(z)|$$

$$\geqslant T_s \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} |F^*_{ij}(z) z^{-D_{ij}}|$$

We will now consider the special case of a lowpass first-order filter. The transfer function will be $F_{ij}(s) = F_i(s) = 1/(1 + sT_{Fi})$, and we will define y_i by $T_{Fi} = y_i T_s$. Expr. 3 now becomes:

$$\left| (z-1)(1 - z^{-1} e^{-1/y_i}) + \frac{w}{y_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} \right| > \frac{w}{y_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij}$$

We apply the maximum-modulus theorem to the region $|z| \geqslant 1$ to show that the critical k_{ij} are obtained by putting $|z| = 1$. Consideration of the roots of this quadratic inequality in z leads us to the sufficient condition that, for all i,

$$\left(\frac{w}{T_s}\right)(y_i T_s) \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} < y_i^2 \frac{(1 - e^{-1/y_i})^2}{(1 + e^{-1/y_i})} \qquad \ldots \qquad (5)$$

Table 1 COMPARISON OF STABILITY CRITERIA

Control system	Result for continuous system	Result for sampled systems (see inequality)
Single-ended, with delays, but no filters	No limit on k_{ij}	4
Single-ended, with delays and filters	$T_{Fi} \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} < \frac{1}{2}$	5
Double-ended, with delays, but no filters	$(\tau_{ij} + d_{ji})_{\max j} \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} < \sqrt{2} - 1$	6
Double-ended, with delays and filters	$\{(\tau_{ij} + d_{ji})_{\max j} + T_{Fi} + T_{Hi}\} \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} < \sqrt{2} - 1$	Not yet known

(*c*) *Double-ended systems with delays but no filters* $[F^*_{ij}(z) = H^*_{ij}(z) = I]$:

Expr. 3 becomes

$$\left|\frac{(z-1)}{T_s} + \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij}(1 + z^{-D_{ji}} z^{-\delta_{ij}})\right| > 2 \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij}$$

$$> \sum_{\substack{i=1 \\ j \neq i}}^{n} k_{ij} |z^{-D_{ij}} + z^{-\delta_{ji}}|$$

Putting $D_i = \{\delta_{ij} + D_{ji}\}_{max\,j,\,j=1,2,\ldots,n,\,\neq 1}$, we may obtain the sufficient condition that

$$T_s D_i \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ij} < \sqrt{\left\{2 + \frac{2}{D_i} + \left(\frac{1}{D_i}\right)^2\right\}} - \left(1 + \frac{1}{D_i}\right) \qquad (6)$$

The double-ended system with filters is somewhat more complicated, and is the subject of current research.

The limiting values of these stability conditions, as $T_s \to 0$, are exactly the same as the results already obtained[1] for the continuous system. This comparison is made in Table 1. These results are of considerable importance where the control system is based on digital logic and the effective sampling period T_s becomes large. Some reduction in gain is essential, in any case, to allow for the discontinuous operation of the control. The reduced gains should still be adequate for proper operation of the system with sampling periods as long as 1 ms.

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M. R. MILLER 26th March 1969

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Feasibility studies of synchronised-oscillator systems for p.c.m. telephone networks

M. R. Miller, M.A., M.Sc.

Abstract

Two important classes of linear control system for pulse-code-modulated exchange-oscillator synchronisation are described. These are known as 'single-ended' and 'double-ended' systems. This paper extends the knowledge of the stability and steady-state frequency for the single-ended system, and derives the corresponding information for the double-ended system. Both systems have discontinuous phase comparators, which can cause undesirably large phase differences in the steady state. This is known as 'wrong-mode operation'. Existence conditions for such a wrong mode are used to detect and eliminate it. The stability criteria include the effects of filters in the error-signal paths. These filters, the line delays and the operating mode will all affect the steady-state frequency of the single-ended system, but not of the double-ended system. The latter is thus more likely to be used in practice. The paper shows that a synchronous integrated p.c.m. network is completely feasible, although the economics of such a system have yet to be proved.

List of symbols

i, j = integer subscripts denoting exchange i and input j
n = number of exchanges in network
t, γ = time, s
$f_i(t)$ = frequency of exchange i at time t, Hz
f_{0i} = frequency of exchange i with no control signals applied, Hz
$\phi_i(t)$ = phase of exchange i at time t, normalised with respect to one frame period
$\phi_{ij}(t)$ = apparent phase difference between i and j, measured at i (in units of normalised phase)
$r_{ij}(t)$ = integer associated with discontinuity in phase detector (in units of normalised phase)
$e_{ij}(t)$ = output of phase detector at i, in line between i and j (in units of normalised phase)
k_{ij} = weighting of error signals derived from link j to i when used at i, Hz per unit phase error
d_{ij} = transmission delay of line from j to i, s
τ_{ij} = transmission delay of data link from j to i, s
$F_{ij}(s)$, $G_{ij}(s)$, $H_{ij}(s)$ = transfer functions of filters in control path from exchange j at exchange i
T_{Fij}, T_{Gij}, T_{Hij} = effective time constants of above filters
$\{e(t)\}_{Gij}$ = signal $e(t)$ after passing through filter with transfer function $G_{ij}(s)$
$A(s)$ = matrix for single-ended system containing connection, delay and filter information
$B(s)$ = similar matrix for double-ended system
δ_{ij} = variation of line delay d_{ij} from ideal, due to temperature variation
$\boldsymbol{\lambda}$ = vector giving operation modes, derived from δ_{ij} and r_{ij} terms

1 Introduction

The use of pulse-code modulation (p.c.m.) for telephone transmission has led to economies of line plant. It is certain that in the future more 'junction' or interexchange telephone circuits will be converted to p.c.m. working, and this suggests that a digital exchange switching p.c.m. messages would be more economic, as well as giving better transmission performance, than a conventional exchange. This would replace, by a single digital process, the demodulators and modulators as well as the audio switching stage which must be used in conventional exchanges. Such a digital exchange has recently been constructed for trials,[1] and one of the problems arising from possible widespread application of such exchanges will be discussed here.

A typical system uses ordinary telephone lines, with digital repeaters in place of the loading coils previously fitted at 2000yd intervals. This allows the use of two pairs of wires for 24 conversations instead of only two; one pair is used for each direction of speech.

To provide an adequate passband of 4kHz for speech, each conversation is sampled at 8kHz. Each sample is quantised and converted to a 7-bit binary code, signifying the voltage applied at the sampling instant. An eighth bit is added to provide interexchange signalling for the channel, and each group of eight bits is allocated a time 'slot' in a 'frame' of 192 bits duration. Thus the samples from 24 conversations are interleaved to effect the required traffic concentration.

Each frame of information corresponds to a single sample of all the input channels, and some means of identifying the start of each frame has to be provided. One method is to introduce fixed patterns into the signalling-bit positions of some frames. Reference to the derived frame-start signal allows identification of each channel for demodulation at the receiving terminal. The detectors provide a 'clock' of the same frequency as the sending-terminal clock, with its phase displaced by the phase shift in the cable, a function of propagation delay.

The line noise below a certain threshold is rejected by the digital system, but the quantising process introduces noise. This can be reduced by a large number of quantising levels and by restricting the number of quantisers used on any given digital connection. One way in which this can be done is to use digital switching.[2] For this, it is necessary that all incoming information be at the same frequency, and it is desirable that the clock phases be identical. Control systems to give these 'homochronous' clocks have been devised, and these introduce a problem of control stability. However, the application of multivariable control theory, in which the clock phases and frequencies are the variables, has led to solutions which show that this approach is feasible. Matrix methods are used in the analysis, to ease the handling of the sets of equations derived from consideration of the frequency of every exchange clock in the network.

A recent paper by Hills[14] uses the same matrix approach, but for large complex networks introduces certain restrictions on the parameters, which are avoided in this paper.

2 Control-system hardware

2.1 Variable-frequency oscillator

The heart of the system is an oscillator whose frequency and phase may be altered by external signals. A moderately stable free-running frequency is required, and this suggests the use of such devices as external phase-shifters as an alternative to variable-reactance components in the tuning circuit. Variable-frequency oscillators may be divided into two classes, one providing continuous, usually linear, adjustment, and the other involving rapid switching between two

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or more distinct frequencies, so that the mean clock rate is that required by the network. The analysis of systems using the latter is somewhat more difficult, owing to the nonlinear nature of the controls.

2.2 Phase detector

As mentioned above, the incoming terminal of a point-to-point p.c.m. system includes equipment for the recovery of the originating clock information. In a switched system, a small buffer store is required in every input to the exchange, to compensate for small changes in line delay due to temperature variations.[2] To write the information into this store, the originating clock must be used, so the same recovery equipment is used at the digital exchanges. The store is read under the control of the exchange clock, so that the amount of information held in the store is a measure of the phase difference between the two exchange clocks. A device in parallel with this buffer store, to indicate the 'fill' of the store, provides a useful form of phase detector at little additional cost. A bistable device which is set by one clock and reset by the other will produce a continuously variable mark/space ratio indicative of the percentage fill of the store, and hence a linear phase difference signal. An alternative is to provide alarms when the store is near the end of its capacity, and to initiate control action at these times only; such a system could also prove satisfactory.

All such detectors are cyclic, since they work within the period of the driving-clock waveforms and take no account of differences of over one cycle. Thus, if the clocks worked to a base of a single frame of 192 bits, the detectors would be correct over this range. A difference of, say, 200 bits would appear as a difference of only eight bits, since a complete cycle of 192 bits would have been ignored. It will be seen later that this cyclic characteristic of phase detectors can lead to an interesting problem of 'wrong-mode operation'.

2.3 Bilateral control

In this paper, we shall only be concerned with systems using bilateral control, wherein the phase of exchange A, if it is directly connected to B, influences, and is influenced by, the phase of B. The preference for this over unilateral systems in which control is not mutual but directed is largely based on security considerations, as a link failure can create a master–slave system.

2.4 Double-ended systems

Systems which use the error signal derived from the store fill at the incoming end only are known as single-ended systems. They suffer from the fact that the line delay varies, and this affects the apparent phase difference when, in fact, the true phase difference may be unchanged. This, in turn, causes the network to operate at a clock rate which is a function of the line delays. To obviate this difficulty, a 'double-ended' system has been proposed[3] in which the error signal is sent to the remote end of the interexchange link over a special path, where it is subtracted from the locally generated store fill. If the two line delays are equal, the effects cancel, and the frequency is unchanged by delay variations. Even if the delays are not always equal, it can be shown that under certain circumstances the effects still cancel.

2.5 Filters

A filter is required at the input to every oscillator, to reduce the effects of high-frequency noise and to smooth the signal from the digital phase detectors described above. This may be a simple first-order lowpass filter. With double-ended systems, it is convenient to use the p.c.m. highway as the path for the store-fill signalling; a single bit every few frames is sufficient. This path will also require a filter to give a continuous output, and, again a first-order lowpass filter is sufficient.

The circuits for clock recovery will not respond instantaneously to a change of frequency, and must be considered as an additional filter in the control path between one oscillator output and the input to another. This may not be a first-order process, but it usually approximates to one.

It will be seen that these filters affect the stability of the system, and it is possible that they will dominate the delays in the criteria for control stability.

2.6 Nonlinear systems

This work is largely concerned with the application of continuous linear control systems to the problem of synchronous operation of the network, but occasional reference is made to nonlinear systems. These are based on the principle that continuous correction is not necessary, and that corrections need only take place when the correct operation of the buffer stores is endangered. The store will absorb any changes of phase difference due to oscillator drift, as well as those due to line delay variations; and it must not become empty nor may it be allowed to overwrite information which has not been read into the exchange. If alarms are provided to detect these conditions, the oscillator frequency can undergo a step change in time to prevent loss of information.

The nonlinear nature of these control systems leads to a discontinuous relation between the network operating frequency and line delays, unless a double-ended system is used.

3 Single-ended systems

A single-ended system will, in practice, always be continuous. Any quantisation of the error signals will cause the network frequency to be one of a number of discrete values,

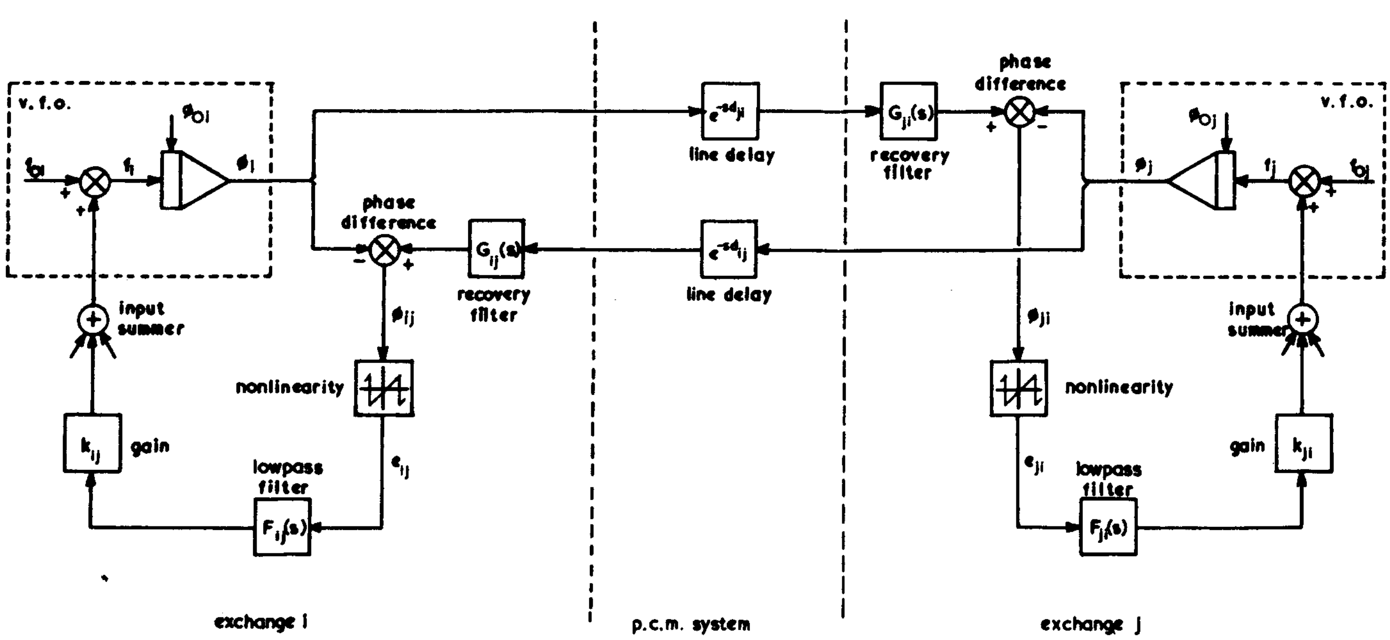

Fig. 1
Single-ended control system

and, under certain circumstances, the control may attempt to follow different values in different parts of the network.

The line delays, which vary with temperature, are normally padded out so that, with the buffer stores half full, the total of line, fixed-pad and buffer-store delays is a multiple of one frame period; this is 125μs with 8kHz sampling. This is the simplest method of satisfying the basic condition that the total delay round any closed loop of the network shall be a multiple of one frame period. This condition is a requirement of the exchange design considered.[2]

The error signal is derived from the clock signals which drive the buffer store. A particular pulse from the incoming line clock is used to set a bistable device, which is reset by a different pulse from the exchange clock supply. These pulses are chosen so that, with the store half full, the output of the bistable device will have a unity mark/space ratio. This condition corresponds to the line delays being at their nominal values and the two exchange clocks being in phase. A lowpass filter is used to smooth the output to a continuous signal, which is then biased to give a zero output from the unity-mark/space-ratio signal.

The signals from all the incoming routes are then added by a summing amplifier whose output is connected to the oscillator, to vary the frequency of the exchange clock so as to minimise the error signals. A block diagram of this process is shown as Fig. 1; both ends of the line are identical, and so the control is bilateral.

The output of the phase comparator at the ith exchange in the line from the jth is

$$e_{ij}(t) = \phi_{ij}(t) + r_{ij}(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $r_{ij}(t)$ is the integer such that

$$r_{ij}(t) \leqslant \tfrac{1}{2} - \phi_{ij}(t) < r_{ij}(t) + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The apparent phase difference is given by

$$\phi_{ij}(t) = \{\phi_{0j}\}_{G_{ij}} + \int_0^{t-d_{ij}} \{f_j(\gamma)\}_{G_{ij}} d\gamma - \phi_{0i} - \int_0^t f_i(\gamma) d\gamma \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The signals $e_{ij}(t)$, after passing through the lowpass filters with transfer functions $F_{ij}(s)$, are amplified and added before being applied to the oscillator input. The gain applied to the signal $e_{ij}(t)$ is k_{ij}, which takes into account the transfer function of the oscillator. Its dimensions are time^{-1} as it is the frequency change resulting from a unit phase error.

We are concerned with operation in a single 'mode' and may treat $r_{ij}(t)$ as constant. The oscillator frequency is therefore given by

$$f_i(t) = f_{0i} + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\{e_{ij}(t)\}_{F_{ij}}$$

Taking Laplace transforms, after substituting for $e_{ij}(t)$, we obtain

$$\Big\{s + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}F_{ij}(s)\Big\} f_i - \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}F_{ij}(s)G_{ij}(s)\exp(-sd_{ij}) f_j$$
$$= f_{0i} + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}F_{ij}(s)\Big\{r_{ij} - \phi_{0i} + \phi_{0j}$$
$$- G_{ij}(s)\Big(\frac{f_{0j}}{s}\Big)(1 - sd_{ij} - \exp(-sd_{ij}))\Big\}$$
$$= f_{0i} + u_i(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where u_i is defined by the above equation. This may be written more conveniently in matrix form:

$$\{s\boldsymbol{I} - \boldsymbol{A}(s)\}\boldsymbol{f} = \boldsymbol{f}_0 + \boldsymbol{u}(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $a_{ii} = -\sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}F_{ij}(s)$

and $\quad a_{ij} = +k_{ij}F_{ij}(s)G_{ij}(s)\exp(-sd_{ij}) \quad i \neq j$

The stability of the system is determined by the terms of the matrix $\boldsymbol{A}(s)$, and the final operating frequency by the components of the vectors $\boldsymbol{u}$ and $\boldsymbol{f}_0$. It will be shown that the final frequency is dependent on the operating mode, delays and filter time constants as well as on the initial frequencies and gains. The stability is a function of the filter time constants and gains only.

The initial phase differences will not affect the final operating frequency, except in so far as the operating mode may be other than that intended by the system design. This is because the integers r_{ij} will be altered.

4 Double-ended system

To overcome the interdependence of operating frequency and delay, an alternative system has been devised.[3] Instead of minimising the apparent phase difference, this

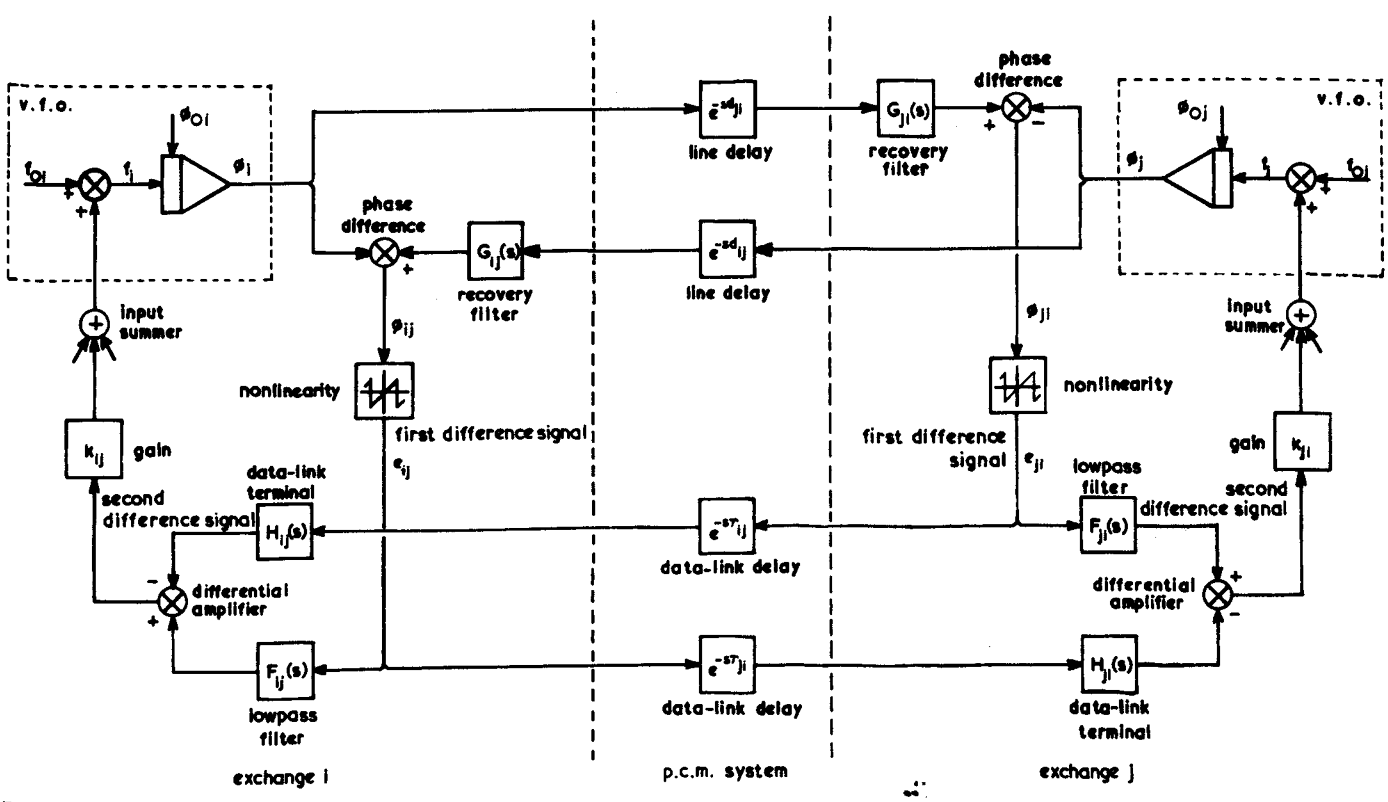

Fig. 2

Double-ended control system

attempts to establish and minimise the true phase difference.

A buffer store is used as before, and the counters are used to derive a 'first-difference' signal $e_{ij}(t)$. This does not control the oscillator directly, but is instead transmitted to the distant exchange. A simple digital link can provide adequate bandwidth if one bit is available every 1·25ms (ten frames). Similarly, the distant signal is transmitted to the local exchange. At each exchange, the distant signal is subtracted from the local signal. The true phase information then adds, and the delay information tends to cancel; complete cancellation will only occur at this stage if the delays are equal. In fact, by arranging that all filters have unity gain at zero frequency, it can be shown that the effect of such inequalities is completely eliminated.

The 'second-difference' signal is passed to the oscillator through a combining network, as before. Fig. 2 shows a block diagram of this process.

In the equation for the exchange frequency, it will be noted that the link for transmitting the 'first-difference' signal has a filter with transfer function $H_{ij}(s)$, and that an additional line delay τ_{ij} is introduced:

$$f_i(t) = f_{0i} + \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}[\{e_{ij}(t)\}_{F_{ij}} - \{e_{ji}(t-\tau_{ij})\}_{H_{ij}}] \quad (6)$$

Again, we take Laplace transforms and write the result in matrix form:

$$\{sI - B(s)\}f = f_0 + v(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where

$$b_{ii} = -\sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\{F_{ij}(s) + H_{ij}(s)G_{ji}(s)\exp(-s\tau_{ij} - sd_{ji})\}$$

$$b_{ij} = +k_{ij}\{F_{ij}(s)G_{ij}(s)\exp(-sd_{ij}) + H_{ij}(s)\exp(-s\tau_{ij})\}$$

$$v_i = \sum_{\substack{j=1 \\ \neq i}}^{n} k_{ij}\Bigg(\{F_{ij}(s)G_{ij}(s) + H_{ij}(s)\}\left[\phi_{0j} + \frac{f_{0j}}{s}\{1 - sd_{ij} - \exp(-sd_{ij})\}\right]$$
$$- \{F_{ij}(s) + H_{ij}(s)G_{ij}(s)\}\left[\phi_{0i} + \frac{f_{0i}}{s}\{1 - sd_{ji} - s\tau_{ij} - \exp(-sd_{ji} - s\tau_{ij})\}\right]$$
$$+ \{r_{ij}F_{ij}(s) - r_{ji}H_{ji}(s)\}\Bigg)$$

Matrix $B(s)$ determines the stability of the system, which is sensitive to the gains, the line delays and the filter time constants. The system operating frequency is independent of operating-mode line delays or filter time constants, and thus the system is considered superior to the single-ended system, despite the apparent disadvantage of a more difficult stability condition. The stability condition gives maximum allowable values for certain components, and any variations within the permitted range do not affect the operating frequency.

5 Operating mode

It has been stated that the output of the phase comparators is cyclic. This allows certain phase states (where all phases are by no means nearly equal) to persist with the control system giving no corrective signal. For example, a ring of three exchanges with linear sawtooth phase detectors of the type described in Section 2.2 could have true relative phases of 0, $\frac{1}{3}$, and $\frac{2}{3}$ frame units, respectively. The exchanges then have true input differences of $+\frac{1}{3}$ and $+\frac{2}{3}$, $-\frac{2}{3}$ and $-\frac{1}{3}$ and $-\frac{1}{3}$ and $+\frac{1}{3}$. This is illustrated in Figs. 3A and B. The comparator outputs at each exchange all sum to zero because of the output discontinuities, so that no control signal is produced. This situation is entirely stable, and is known as wrong-mode operation.[4]

There are two basic conditions for a wrong mode, and both must be satisfied. The first is that the sum of the true phase differences around any loop of exchanges must be a multiple of one frame unit, where this is the period of the cyclicity. The second condition is that at every exchange there must be no control signal, and therefore that the sum of the phase differences, modulo one frame unit, must be zero. A solution which gives the operating mode on a boundary between two linear parts of the comparator output is not stable; thus there can be no wrong modes for four bilaterally connected exchanges, and other special cases have a reduced number of modes.

A matrix approach to this problem has been suggested.[5] This was limited in its application to the case of three exchanges, and the results of investigations into higher-order cases were subsequently published.[6] This second paper noted that, for a wrong mode to exist, at least one phase comparator must have passed through a discontinuity; this implies that, with respect to a reference oscillator, at least one exchange must have a true phase outside the range $-\frac{1}{4}$ to $+\frac{1}{4}$. If this state could be recognised, the network could be made to disconnect enough control paths to break the loop creating the unwanted condition.

Use can be made of the hierarchic nature of a practical telephone network to designate a phase-reference 'master' exchange. This can then send signals to every exchange in the network, over the minimum of intervening links, to indicate the true phase of all the exchanges. If any phase exceeds the allowable range for more than a short time, an alarm will be generated.[7] At every exchange, there will be a means to indicate the parent, or a substitute in the case of failure of the parent or the link to it. On receipt of the alarm signal, all control signals, except that from the parent, will be ignored. The network is thus reduced to a master–slave control until such time as the alarm condition is removed. Security measures will be included to prevent false operation of the alarm system.

An alternative alarm device makes use of the limited

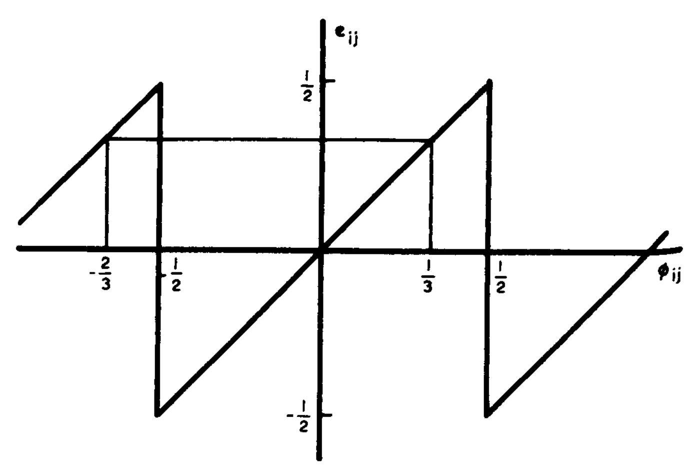

Fig. 3A

Phase comparator output

If $\phi_{ij} = -2/3$, $e_{ij} = +1/3$

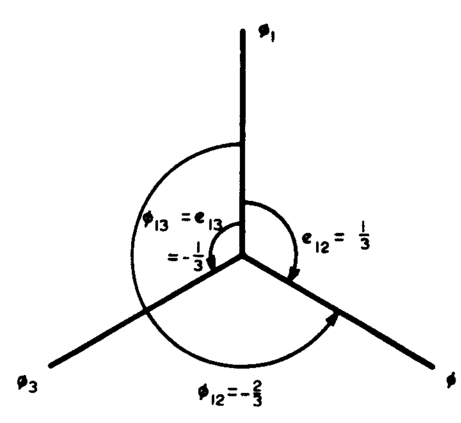

Fig. 3B

Phases of three exchanges in wrong mode

Note that $e_{12} + e_{13} = 0$

number of links connecting any given exchange to a 'master' exchange. In a typical network, every exchange will be connected to a given main trunk exchange by not more than four links. If the exchange has a true phase difference of more than $\frac{1}{4}$ with respect to this central exchange, at least one link must have over $\frac{1}{16}$ phase difference between its two ends. In the system we have discussed, this is a difference of $1\frac{1}{2}$ slots of 8 bits, or three times the normal temperature variation (see Section 2.1). In more dispersed networks, this alternative method would not give an adequate alarm, but, under the above circumstances, the alarm would function.

A wrong mode will arise either from a network being set up incorrectly, or as a result of a transient excursion. Such transients could be the result of noise at an input or as a consequence of adding or removing a link, for example. To avoid wrong modes, care must first be taken to ensure that the phase differences on setting up the network are such that the correct mode will result. A study of transient behaviour[12] enables us to determine bounds on the system parameters, so that the correct mode is never left in the event of a component breakdown.

It is thus possible to both detect and eliminate any wrong-mode condition that could arise. It is considered important that the network be normally highly connected so as to give the tightest possible control. It must be remembered also that, for a wrong operating mode to be set up, a phase difference far greater than that which can be accepted by the buffer stores will have occurred. We will later consider the means by which such phase excursions can be reduced to protect the information in the buffer stores; this will also tend to eliminate the possibility of wrong-mode operation occurring except after initial setup or wide-scale fault conditions.

6 System operating frequency

It is a requirement of the control system that a highly stable operating frequency be produced. The basic oscillators will be of a moderate-stability design, and the injection of control signals will be arranged in such a way that this basic stability is unimpaired. We thus need to know the action of the control system on the network frequency, with a view to the optimisation of the controller.

We may obtain the network frequency by the application of the Laplace final-value theorem to the transformed equations; this is only valid if the system is stable, which we assume at this stage, and in the region of validity of the transformations, which is for conditions in the vicinity of a stable operating point. The steady-state values of all the frequency variables will be the same, so we may use Cramer's rule to obtain the transform of any one of these variables, and to apply the Laplace theorem to this transformer.

We will derive the expression (eqn. 11) for the final frequency of the double-ended system; the expression (eqn. 13) for the single-ended system follows by neglecting the components of the error data link, $H_{ij}(s)$ and τ_{ij}.

We may decompose the matrix functions of s into a number of other matrices which are independent of s. For example,

$$B(s) = B_0 + sB_1 + s^2B_2 + \dots$$

where $B_0 = B(0)$.

The filter transfer functions can also be expressed as power series, e.g.

$$F_{ij}(s) = F_{ij}(0) - sT_{Fij} + \text{higher powers of } s$$

When $F_{ij}(s)$ represents a first-order lowpass filter, T_{Fij} is the time constant of the filter. For higher-order filters, a reference to time constants implies these equivalent constants.

When using Cramer's rule, we use determinants of matrices which are derived from the system equations by replacing, say, the ξth column of $\{sI - B(s)\}$ in eqn. 7 by the column vector of the right-hand side, namely, $f_0 + v(s)$. We will denote this matrix by $\beta_\xi(s)$, and decompose this as above for each $\xi = 1, 2, \dots n$.

The final value of this ξth frequency element is

$$\lim_{t\to\infty} f_\xi(t) = f_s \equiv \lim_{s\to 0} s\frac{\det|\beta_\xi(s)|}{\det|sI - B(s)|} \quad . \quad . \quad . \quad (8)$$

where the determinants are of the matrices defined above. This result is quoted by other authors, but here we will reduce it to a simplified form for a special case of particular interest.

The denominator must have no term in s^0 for the frequency to be non-zero. This is assured by the form of the system equations for which $\det|B(0)| = 0$.

We will denote the cofactor of the element b_{xij} in B_x by B_{xij}. The expansion for f_s now becomes

$$f_s = \frac{\det|\beta_{0\xi}|}{\frac{1}{s}\det|sI - B(s)|}$$

$$= \frac{\sum_{i=1}^{n} B_{0i\xi}\{f_{0i} + v_i(0)\}}{\sum_{i=1}^{n} B_{0ii} - \sum_{i=1}^{n}\sum_{j=1}^{n} b_{1ij}B_{0ij}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For the special case when each input to an exchange has a common gain, we may apply the theorem given in Appendix 13 to the evaluation of the cofactors B_{0ij}. The gains are then

$$k_{ij} = k_i c_{ij}$$

where the connection factors c_{ij} and c_{ji} are equal, taking the value 1 when the ith and jth exchanges have a direct link, and zero otherwise. We assume that all $F_{ij}(0) = F$; G and H are similarly defined. With these restrictions, we show that $k_i B_{0ij}$ is constant for all $i, j = 1, 2, \dots n$.

Inserting the matrix elements and cofactors in eqn. 9, we obtain

$$f_s = \frac{\sum_{i=1}^{n}(f_{0i}/k_i) + (F-H)\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} + (F-H)(G-1)\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\phi_{0i}}{\sum_{i=1}^{n}(1/k_i) + \sum_{i=1}^{n}\sum_{j=1}^{n}\{(G-1)(T_{Fij} - T_{Hij} - H\tau_{ij}) + (F-H)(T_{Gij} + Gd_{ij})\}} \quad . \quad (10)$$

We note that $c_{ii} = 0$ in this expression. The recovery circuits are such that $G = 1$; putting $H = 0$, we obtain the same expression as would be produced by this operation on the single-ended-system eqns. 5.

The network frequency of the double-ended system is given by

$$f_s = \frac{\sum_{i=1}^{n}(f_{0i}/k_i) + (F-H)\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}}{\sum_{i=1}^{n}(1/k_i) + (F-H)\sum_{i=1}^{n}\sum_{j=1}^{n}(T_{Gij} + d_{ij})} \quad . \quad (11)$$

The system will usually be balanced, with $F = H = 1$, so that eqn. 11 reduces to

$$f_s = \frac{\sum_{i=1}^{n}(f_{0i}/k_i)}{\sum_{i=1}^{n}(1/k_i)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The single-ended system has $H = 0$, and, without loss of generality, $F = 1$. Eqn. 11 now becomes

$$f_s = \frac{\sum_{i=1}^{n}(f_{0i}/k_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}}{\sum_{i=1}^{n}(1/k_i) + \sum_{i=1}^{n}\sum_{j=1}^{n}(T_{Gij} + d_{ij})} \quad . \quad . \quad . \quad . \quad (13)$$

We note that the delays are usually padded out to integral numbers of frames, and the integers r_{ij} correspond, in the correct operating mode, to these numbers. Any variation of the delay d_{ij} from this ideal is denoted by δ_{ij}, and λ_v is an element of the $(n-1)$-dimensional vector $\boldsymbol{\lambda}$ which defines the operating mode;[5,6] this vector is zero for the desired in-phase mode, and the sum of its elements is zero for all other modes only when the δ_{ij} are zero. We insert these values in eqn. 13 to obtain

$$f_s = \frac{\sum_{i=1}^{n}(f_{0i}/k_i) + \sum_{v=1}^{n-1}\lambda_v}{\sum_{i=1}^{n}(1/k_i) + \sum_{i=1}^{n}\sum_{j=1}^{n}(\delta_{ij} + T_{Gij})} \quad . \quad . \quad . \quad . \quad (14)$$

Thus any variation of line delays or operating mode will have a considerable effect on the system frequency with the single-ended method of control. This is undesirable and may well put this frequency outside the controllable range of the oscillators. Thus the double-ended method of control is more likely to be used in practice.

Hills,[14] in an appendix, points out this variation of operating frequency with the operating mode, with special reference to two exchanges linked together. For this case, with delays in the lines, the two possible modes lead to two possible frequencies. The effect of delay on operating modes was discussed by the present author in Reference 5. The operating mode of any network will depend on the phase differences at the moment of connection of the controls.

A method that has been proposed[15] to allow some control of the operating frequency includes an integrating device in the frequency-control path at all but one exchange. This makes all the frequencies follow that of the exchange with no integrator. Should this exchange fail, the network will become unstable unless an integrator at some other exchange is disconnected. Such a 'master' exchange would normally be an international entry point to the system, and would have a permanent supervisory staff. The dynamic conditions of the system should not be affected by the addition of the integrators, provided that the time constants are long enough.

7 Stability of linear systems

The general system equations are of the form

$$\{s\boldsymbol{I} - \boldsymbol{A}(s)\}\,\bar{\boldsymbol{f}} = \boldsymbol{c}$$

where the vector $\boldsymbol{c}$ determines the final system frequency and is a function of the initial conditions. The characteristic equation determining stability is

$$\det|s\boldsymbol{I} - \boldsymbol{A}(s)| = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

7.1 Systems without filters

For systems with no delays or filters, the matrix A is, in fact, independent of the Laplace variable s. Thus the stability is determined by the eigenvalues of the matrix A. Early work in this direction established that such systems are always stable,[8] whether they are double- or single-ended.

The inclusion of time delays alone is more difficult. We note that, if the roots in $s = \sigma + j\omega$ of eqn. 15 have positive real parts ($\sigma > 0$), the system is unstable. Thus the matrix $\boldsymbol{Q}(s) = |s\boldsymbol{I} - \boldsymbol{A}(s)|$ may not be singular for $\sigma \geqslant 0$ and $j\omega \neq 0$, if the system is to be stable. If for some $s = s_1$ this is singular, $\lambda = 0$ is a root of

$$\det|\lambda\boldsymbol{I} - \boldsymbol{Q}(s_1)| = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

We now use the Gershgorin circle theorem[9,10] to locate the eigenvalues λ of the matrix $\boldsymbol{Q}(s_1)$ from eqn. 16. This states that the eigenvalues lie in that region of the complex plane which is the union of the circles whose centres are $q_{ii}(s_1)$ and whose radii are

$$\sum_{\substack{j=1\\ \neq i}}^{n}|q_{ij}(s_1)|$$

$i = 1, 2, \ldots n$, and where $q_{ij}(s_1)$ is the i, jth element of $\boldsymbol{Q}(s_1)$. These circles cannot contain the origin, whence $\lambda \neq 0$, if, for all i,

$$|q_{ii}(s_1)| > \sum_{j=1}^{n}|q_{ij}(s_1)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If this inequality is satisfied for s_1 anywhere in the right-hand half of the s plane, the system with characteristic eqn. 15 is stable. This application of Gershgorin's theorem has been called the 'diagonal dominance theorem.[11,17]

We examine two cases:

(a) *Single-ended system described by eqn. 4, without filters:*[16]

$$q_{ii}(s) = s + \sum_{\substack{j=1\\ \neq i}}^{n} k_{ji}$$

$$q_{ij}(s) = -\,k_{ji}\exp(-sd_{ji})$$

(b) *Double-ended system described by eqn. 7, without filters:*

$$q_{ii}(s) = s + \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\{1 + \exp(-s\tau_{ij} - sd_{ji})\}$$

$$q_{ij}(s) = -\,k_{ij}\{\exp(-sd_{ij}) + \exp(-s\tau_{ij})\}$$

We now examine the possibility that $\boldsymbol{Q}(s)$ is singular for $s = \sigma + j\omega$ in the right-hand halfplane. For $\sigma > 0$, $|\exp(-sd_{ij})| < 1$, so that

$$\left.\begin{aligned}\sum_{\substack{j=1\\ \neq i}}^{n}|k_{ij}\exp(-sd_{ji})| &= \sum_{\substack{j=1\\ \neq i}}^{n}k_{ji}|\exp(-sd_{ji})|\\ &< \sum_{\substack{j=1\\ \neq i}}^{n}k_{ji}\end{aligned}\right\} \quad . \quad (18)$$

In case (a) the system is stable, from expr. 17, provided that' for all $\sigma \geqslant 0$ and $s \neq 0$,

$$\left|s + \sum_{\substack{j=1\\ j\neq j}}^{n}k_{ji}\right| > \sum_{\substack{j=1\\ \neq i}}^{n}k_{ji} \geqslant \sum_{\substack{j=1\\ \neq i}}^{n}|k_{ji}\exp(-sd_{ji})| \quad . \quad (19)$$

Expr. 19 is always satisfied. There is a root at $s = 0$, corresponding to the lack of a reference frequency. This result

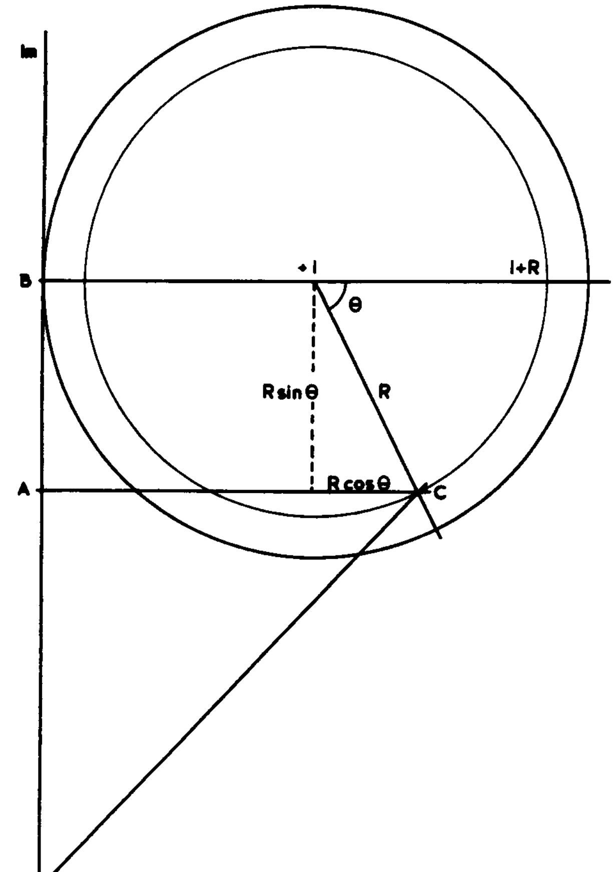

Fig. 4

Stability diagram for double-ended system

$AB = R\sin\theta$
$OC = 2$
$OB = \dfrac{\omega}{\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}}$

extends to the case of general values of k_{ij}, the result deduced by West,[11] which is that this system is stable for any network configuration in the absence of filters.

For case (b), in the absence of filters,

$$\left| s + \sum_{\substack{j=1\\j\neq i}}^{n} k_{ij}\{1 + \exp(-s\tau_{ij} - sd_{ji})\}\right| > 2\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij} > \sum_{\substack{j=1\\ \neq i}}^{n} \left| -k_{ij}\exp(-sd_{ij}) - k_{ij}\exp(-s\tau_{ij})\right|$$

The worst case is represented by $s = j\omega$ when, for stability,

$$\left| \frac{j\omega}{\sum_{\substack{j=1\\j\neq i}}^{n} k_{ji}} + \frac{\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\exp\{-j\omega(\tau_{ij} + d_{ji})\}}{\sum_{\substack{j=1\\ \neq i}}^{n} k_{ji}} + 1 \right| > 2 \quad (20)$$

The term

$$\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\exp\{-j\omega(\tau_{ij} + d_{ji})\}$$

has modulus $M \leqslant \sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}$ and argument $\theta < \omega(\tau_{ij} + d_{ji})_{max\,j}$,

where $j = 1, 2, \ldots n$.

From Fig. 4 we deduce the condition

$$\frac{\omega}{\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}} > R\sin\theta + \sqrt{\{4 - (1 + R\cos\theta)^2\}}$$

where

$$R = \frac{M}{\sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij}} \leqslant 1$$

Putting $R = 1$ and $\theta = \omega(\tau_{ij} + d_{ji})_{max\,j} = \omega D_{iM}$, we obtain

$$\frac{1}{D_{iM}\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}} > \frac{1}{\theta}[\sin\theta + \sqrt{\{4 - (1 + \cos\theta)^2\}}]_{max\,\theta}$$

and the right-hand expression equals $1 + \sqrt{2}$.

Thus, for stability, the parameters must satisfy the sufficient condition that, for all i,

$$(\tau_{ij} + d_{ji})_{max\,j}\left(\sum_{\substack{j=1\\ \neq i}}^{n} k_{ji}\right) < \sqrt{2} - 1 \quad . \quad . \quad . \quad . \quad (21)$$

This is an improvement on criteria deduced from cases 5 and 6 of Reference 8.

7.2 Systems with filters

We again consider the single- and double-ended systems, but with the filter terms of eqns. 4 and 7, respectively.

For the single-ended system, expr. 17 becomes

$$\left| s + \sum_{j=1}^{n} k_{ij}F_{ij}(s)\right| > \sum_{j=1}^{n} |k_{ij}F_{ij}(s)G_{ij}(s)\exp(sd_{ij})|$$

and, if the system is stable, this must be satisfied for all values of s in the right-hand halfplane. As in eqn. 18, we eliminate the delay terms; and, in addition, we will make the simplifying assumption that, for all j, the filters $G_{ij}(s) = G_i(s)$. The condition for stability now becomes

$$\left| s + \sum_{j=1}^{n} k_{ij}F_{ij}(s)\right| > |G_i(s)| \sum_{j=1}^{n} k_{ij}\,|F_{ij}(s)|$$

We now assume that, for all j, the filters $F_{ji}(s) = F_i(s) = 1/(1 + sT_{Fi})$. The stability condition now becomes

$$|G_i(s)| \cdot \left| \frac{1/s}{1/s + (1 + sT_{Fi})/\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}} \right| < 1 \quad . \quad . \quad . \quad (22)$$

This may be satisfied in one of several ways. We have assumed a first-order form of the filter transfer function $F_i(s)$, and have found that this creates a second-order transfer function in the second term of eqn. 22; thus resonances are possible. The filter $G_i(s)$ could also have a resonance. A sufficient condition for the stability is for neither filter to have a resonance. Alternatively, one could have a resonance which is damped by the other, nonresonant, filter. This latter condition is also a sufficient condition, as is implicit in the method used. The stability condition for no resonance of either filter is that, for all s in the left-hand halfplane,

$$|G_i(s)| < 1 \text{ and } T_{Fi}\left(\sum_{j=1}^{n} k_{ij}\right) < \tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad (23)$$

For the double-ended system, the situation is as before, but complicated by the appearance on the diagonal of delay terms corresponding to the additional feedback loop used for comparison of store fills. With the removal of the delay terms on the right-hand side of expr. 17, this becomes

$$\left| s + \sum_{j=1}^{n} k_{ij}[F_{ij}(s) + G_{ij}(s)H_{ij}(s)\exp\{-s(\tau_{ij} + d_{ji})\}]\right| > \sum_{j=1}^{n} k_{ij}|F_{ij}(s)G_{ij}(s) + H_{ij}(s)| \quad (24)$$

If we can assume that all the filters are nonresonant, we can show that the right-hand side of expr. 24 is less than

$$2\sum_{j=1}^{n} k_{ij}$$

for all s in the right-hand halfplane which we are examining.

We again assume a first-order form for the filters, e.g.

$$F_{ij}(s) = \frac{1}{1 + sT_{Fij}}$$

and similarly for G_{ij} and H_{ij}. We also put

$$T_{FiM} = (T_{Fij})_{max\,j}$$

for $j = 1, 2, \ldots n$, and define T_{GiM} and T_{HiM} similarly. Finally, we define D_{iM} as $(\tau_{ij} + d_{ji})_{max\,j}$. By an extension of the method used to derive expr. 21, it is possible to show that the system will be stable if, for all i,

$$(T_{FiM} + T_{GiM} + T_{HiM} + D_{iM})\sum_{j=1}^{n} k_{ij} < \sqrt{2} - 1 \quad (25)$$

This general condition now allows us to be confident of the stable operation of the system. A more exact condition could be obtained, particularly for given shapes of network, but these are unlikely to give a great improvement of operating conditions.

It will be found in practice that the filter time constants are larger than the line delays, and thus determine the stability of the system. This will lessen the differential between the single- and double-ended systems. Possible magnitudes of the time constants and delays are D_{iM} and $T_{GiM} \simeq 10^{-4}$s, and T_{HiM} and $T_{FiM} \simeq 10^{-2}$s. The latter will obviously dominate the system. The resultant system gains for either system will be of a similar order of magnitude, and will be sufficient to allow the network to settle to a common frequency with steady-state phase differences that are within the capacity of the buffer stores.

8 Design of buffer stores

The buffer stores are provided initially to absorb the changes in line delays as the temperature changes. Where symmetrical-pair cables are used, the annual temperature variations are sufficient to cause delay changes of about 1% of the total delay. With lengths of up to 50 miles, storage for about 6 digits must be provided in the buffers to absorb this variation. For longer lines, coaxial cables are used which have a relatively small temperature coefficient of delay.

In addition to this, the phase differences between oscillators will also require buffer storage. We thus need to have realistic estimates of the transients in the system, to establish the optimum size of the buffer stores. Methods based on Lyapunov functions are available[12] which show that transients decay faster in a tightly connected network than in a widely dispersed one. The magnitude of the expected transients is also inversely proportional to the control-loop gains, which, in turn, are inversely proportional to the line delays and time constants of the system, if the stability criteria are to be satisfied. Thus a further function of the line delays appears in the expression for finding the size of the stores.

It has already been stated that, in 'wrong-mode' operation of the system, some very large, steady-state phase differences may appear. This will put many buffer stores outside their available range, and traffic will be lost, so that means for the detection and elimination of these wrong modes have to be included in the system hardware.

9 Nonlinear systems

Most studies have been concerned with linear systems having a sawtooth nonlinearity in the phase detectors. All such systems are likely to have saturation nonlinearities in them, and this will affect the stability and operation. The problem of stability can be studied by means of Lyapunov functions for the state space. Alternatively, the state space can be considered logically, and some concept of stability can be obtained. This has been done for an interesting class of nonlinearity proposed by Duerdoth.[13]

Duerdoth proposed a system in which a 3-state output was obtained from the phase detectors. These indicated whether the phase difference was large and positive, large and negative, or negligible. In the last instance, no control action is needed, but the first two require immediate control action to bring the phase differences within the capacity of the buffer stores. An additive combination law is not acceptable for the outputs of the comparators, since action can take place in such a way as to worsen the situation rather than improve it. To remedy this, no action is taken in the event of both positive and negative outputs being present. In the correct mode of operation, there will be one exchange where only one type of signal is present, and this will take action to remove these signals. This then allows some other exchange to take action, and the control will be sequential.

One interesting effect of these nonlinearities is the way in which wrong-mode operation takes place. As the output is not linearly related to the true phase difference, the stable points found for the quasilinear system are modified to become regions of the state space in which no control action takes place; the reason is, as before, the addition of several nonzero control signals to a zero-action signal at the oscillator input. The precise manner in which these modes are modified depends on the type of nonlinearity concerned.

The potential advantages of the nonlinear systems proposed by Duerdoth are that the control circuits can be built from the same type of hardware as the rest of the exchange, and it is claimed that the provision of fail-safe operation is eased. Analysis of this type of system is difficult, and little is known about the operation of general networks using them.

10 Conclusions

At this stage, some detail work still remains for the two main types of system design discussed in this paper. However, we may now state with confidence that it is quite possible to build a stable, synchronous integrated p.c.m. network. The most profitable lines of further study now seem to be the effects of the discontinuous 'sampled' nature of the system[18] (neglected in this paper), the effect of nonlinearities and the magnitude of the transients.

The economic merits of synchronous, as opposed to asynchronous, systems have yet to be proved, and it is to provide a basis for this study that the work contained in this paper must be continued.

It is felt that this information will be invaluable to system planners at this early stage.

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13 Appendix

The following theorem allows us to deduce the expression for the final frequency in eqn. 9.

Theorem:

The cofactors A_{0qr} of the element a_{0qr} in $\det |A(0)|$ are related by

$$k_q A_{0qr} = \text{constant for all } q \text{ and } r$$

where $a_{0qr} = k_q c_{qr}$ for $q \neq r$

$$a_{0qq} = -k_q \sum_{\substack{r=1 \\ r \neq q}}^{n} c_{qr}$$

and $c_{qr} = c_{rq} = 1$ when the qth and rth stations are directly connected, and zero when they are not.

Proof:

From the above conditions, it follows that

$$\sum_{r=1}^{n} a_{0qr} = 0 \text{ and } \sum_{q=1}^{n} (a_{0qr}/k_q) = 0$$

Consider now the expansion of

$$\det |a_{ij}| = \det | s\delta_{ij} - a_{0ij} |$$

where δ_{ij} is the Kronecker delta. Adding all the columns to the rth column,

$$\det |a_{ij}| = s \left| \begin{array}{c:c:c} & 1 & \\ & 1 & \\ & 1 & \\ & 1 & \\ & 1 & \\ & 1 & \end{array} \right|$$

$$\begin{array}{ccc} r-1 & r\text{th} & n-r \end{array}$$

Dividing the ith row by k_i, for all i,

$$\det|a_{ij}| = s\prod_{i=1}^{n} k_i \left|\begin{array}{c:c:c} \dfrac{a_{ij}}{k_i} & \dfrac{1}{k_i} & \dfrac{a_{ij}}{k_i} \end{array}\right|$$

Adding all the rows to the qth row,

$$\det|a_{ij}| = s\prod_{i=1}^{n} k_i \left|\begin{array}{c:c:c} \dfrac{a_{ij}}{k_i} & \dfrac{1}{k_i} & \dfrac{a_{ij}}{k_i} \\ \hdashline \dfrac{s}{k_j} & \dfrac{1}{\Sigma k_i} & \dfrac{s}{k_j} \\ \hdashline \dfrac{a_{ij}}{k_i} & \dfrac{1}{k_i} & \dfrac{a_{ij}}{k_i} \end{array}\right| \begin{array}{l} q-1 \\ \\ q\text{th} \\ \\ n-q \end{array}$$

Multiplying the ith row by k_i, for all i,

$$\det|a_{ij}| = s\left|\begin{array}{c:c:c} a_{ij} & 1 & a_{ij} \\ \hdashline s\dfrac{k_q}{k_j} & k_q\Sigma\dfrac{1}{k_i} & s\dfrac{k_q}{k_j} \\ \hdashline a_{ij} & 1 & a_{ij} \end{array}\right|$$

We may regard this expansion as a power series in s. The coefficient of s is given by

$$k_q\sum_{i=1}^{n}(1/k_i)A_{0qr}$$

for all q and r.

Since this does not depend on the method of evaluation, k_qA_{0qr} is constant for all q and r.

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Results of some Studies of p.c.m. Network Synchronisation Systems*

Résultats de certaines études des systèmes de synchronisation de réseaux à modulation de code d'impulsions

Ergebnisse einiger Studien an PCM-Netzwerksynchronisationssystemen

Результаты некоторых трудов по системам синхронизации импульсно-модулируемых сетей

P. C. PARKS† and M. R. MILLER‡

Pulse code modulation is used for telephone transmission. Now experimental digital exchanges are being built to switch the coded speech data. Problems of clock oscillator synchronisation, arising from the interconnection of such exchanges, are investigated.

Summary—For synchronised operation of p.c.m. telephone exchanges it is desired to ensure equality of oscillator frequency at every exchange. A multivariable control system using a phase-locked technique can be devised to produce long-term synchronisation.

Many authors have been active in this field, and this paper indicates some of the results of analysis of the stability and operating frequency, with particular reference to work of the present authors. An important system, known as the "double-ended system", has not been treated by most previous authors, although this is likely to form the basis of a practical system. This is described here and details of its performance are given.

The performance criterion for the system is that no two connected exchanges shall have a phase difference greater than a certain fixed amount. Certain operation modes of the system violate this condition and they are investigated. Means for eliminating them are suggested. The transient motion may be examined by use of Liapunov functions.

INTRODUCTION

PULSE code modulation (p.c.m.) is being introduced into telephone transmission systems as a means of increasing the traffic-carrying capacity of the existing cables. At present these systems are being used on inter-exchange routes in the U.K. and elsewhere.

A typical system enables twenty-four telephone calls to be transmitted over two pairs of wires. Each speech channel is sampled at 8 kHz, to allow transmission of frequencies of up to 4 kHz, and the samples are interleaved with those from the other twenty-three channels to achieve the required concentration of the traffic. Each sample is then quantised and encoded in binary form, seven bits being used for this purpose. An eighth bit is added to permit signalling between the exchanges. The resulting 1·536 Mb/s pulse stream is compatible with the use of digital repeaters every 2000 yds (1830 m) on ordinary telephone lines.

A group of 192 bits is called a frame, and corresponds to a single sample of every input channel. Information is added to the frame to identify the start of the first channel "slot", and all others are identified by reference to this. Counters and detecting logic are used to recover this information at the incoming terminal so that demodulation may take place.

The digital message is immune to line noise below a certain threshold, at which point it becomes severely mutilated. Modulation noise is introduced by the quantising process, so the quality of the signals depends on both the number of quantising levels and on the number of times the operation is performed in a connection using several p.c.m. links. To improve the quality it is suggested [1] that when the signals are switched, this should be in digital form to avoid unnecessary distortion.

To perform the switching function properly it is necessary that all p.c.m. links connected to an exchange shall use a common sampling (and frame) repetition frequency. It is also desirable, but not

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essential, that the phases be identical, depending on the location of certain delay elements within the exchange [2]. The pulses representing the start of each "frame" would then be received from each link simultaneously.

In this paper a system is discussed wherein all signals entering the exchange are required to be in phase at all times, that is that the signals shall be "homochronous". Any departure from this ideal state will result in loss of information or the generation of spurious signals, and is thus to be avoided.

Line propagation delays will cause phase shifts and this will result in a requirement for additional information storage within the exchange unless the line delays are arranged so that the total delay round any closed loop is a multiple of the period of one frame (here 125μs). Such a loop could be the "go" and "return" paths between exchanges A and B, i.e. A–B–A, or a more complicated path such as A–B–C–D–A. One solution is to pad out each line so that it is a multiple of 125μs, which is the delay of about 17 miles (27 km) of cable.

The line delays, however, vary with temperature, so it is proposed to compensate for this variation by inserting a small "buffer store" in every input to the exchange. This would provide a complementary delay variation.

One form of this buffer store is a number of bistable devices which are addressed by counters locked to the frame-start signals in the message stream. If the appropriate line pulse is of value 1 the bistable element is set, otherwise it is reset. Reading the state of the store elements is under the control of similar counters locked to the local exchange clock waveform. The time for which information is stored in a given element is the instantaneous value of the buffer storage delay. This will be a function of the line delay and of the true phase difference between the distant exchange clock, which generated the incoming signals, and the local exchange clock which controls the switching.

Within the period of the store control counters, a signal indicating "store fill" can be derived. This period is typically of length one frame. One performance criterion is the minimisation of this store fill, and hence the size of the store, since these stores are fitted to every line in the network.

Since the line delay affects the store fill signal, a simple "single-ended" control system to minimise the store fill will operate all the clocks at a common frequency which will be a function of all the delays in the system. As this has to be within the controllable range of all the oscillators, a variation of this system is proposed [3] to eliminate the effects of line delays.

The value of the fill signal from an incoming line buffer store is transmitted back to the exchange at the distant end of that line. At this distant exchange the fill signal is subtracted from the locally generated store fill signal from the same route. The result is a function of the true phase difference and the difference, usually small, in the delays of the two directions of transmission. It will later be shown that this "double-ended" system can produce a common clock frequency independent of the actual line delays in the network, even if the delays in the two directions of transmission of a given link are unequal.

The operation of both the single-ended and the double-ended systems will now be discussed.

NOTATION

i, j	integer subscripts denoting exchange i, input j
n	number of exchanges in network
t, τ	time (sec)
$f_i(t)$	frequency of exchange i at time t (Hz)
f_{oi}	frequency of exchange i with no control signals applied (Hz)
$\phi_i(t)$	phase of exchange i at time t, normalised with respect to one frame period
$\phi_{ij}(t)$	apparent phase difference between i & j, measured at i (in units of normalised phase)
$e_{ij}(t)$	output of phase detector at i terminating line from j to i (in units of normalised phase)
x_i	phase of i relative to a reference phase, normalised with respect to one frame period
k_{ij}	weighting of error signals derived from link j to i when used at i (Hz/unit phase error)
d_{ij}	transmission delay of line from j to i (sec)
$r_{ij}(t)$	integer associated with discontinuity in phase detector for $\phi_j - \phi_i$
N_{ij}	number of "frames" of information stored in line from j to i
$q_{ij}(t) = r_{ij}(t) - N_{ij}$,	and constant within linear range of comparator
$F_{ij}(s)$, $G_{ij}(s)$, $H_{ij}(s)$	filters at exchange i in control path from exchange j
$\mathbf{A}(s)$	matrix containing connection, delay and filter information
$\mathbf{A}^*$	matrix in phase difference equations derived from $\mathbf{B}$ and $\mathbf{A}$
$\mathbf{A}$	matrix of connection factors a_{ij} with a_{ii} such that $\sum_{j=1}^{n} a_{ij} = 0$
$\mathbf{B}(s)$	matrix similar to $\mathbf{A}(s)$ for the "double-ended" system
$\mathbf{B}$	matrix for deriving phase differences
$\mathbf{D}, \mathbf{D}^*$	matrices as $\mathbf{A}$, $\mathbf{A}^*$ containing line delays
$\mathbf{\Delta}$	matrix of variations δ_{ij} of line delays d_{ij} from ideal values
$\boldsymbol{\lambda}$	vector giving operation modes, derived from matrix $\mathbf{Q} = \{q_{ij}\}$

SYSTEM EQUATIONS

The buffer stores

The fill of the buffer store, defined by the number of stored digits awaiting reading, is equal to the apparent phase difference across the store. The phases are normalised with respect to the sampling period of one frame (125 μs), as the sampling frequency is the lowest frequency in the system. The control logic of the stores consists of counters which reset to zero on the 192nd pulse, so the apparent phase difference or store fill, $e_{ij}(t)$, is always in the range $-\frac{1}{2}$ to $+\frac{1}{2}$.

The fill of the store at the ith exchange, terminating the line from the jth exchange, is given by

$$e_{ij}(t)=\phi_{ij}(t)+r_{ij}(t) \tag{1}$$

where

$$r_{ij}(t)=[\tfrac{1}{2}-\phi_{ij}(t)] \tag{2}$$

using the Gaussian bracket notation that $[x]$ is the integer such that $[x]\leqslant x<[x+1]$ (see Fig. 1). When taking Laplace transforms, r_{ij} is assumed to be constant, i.e. the transform is only valid within a zone of the phase-space for which no comparators pass through a discontinuity.

We use $(f_j)_{Gij}$ to denote the frequency f_j after it has passed through the clock recovery filter whose effective transfer function is $G_{ij}(s)$. The apparent phase difference is given by

$$\begin{aligned}\phi_{ij}(t)&=\{\phi_j(t-d_{ij})\}_{Gij}-\phi_i(t)\\&=\{\phi_j(t)\}_{Gij}-\int_{t-d_{ij}}^{t}\{f_j(\tau)\}_{Gij}\,.\,\mathrm{d}\tau-\phi_i(t)\end{aligned} \tag{3}$$

where d_{ij} is the propagation delay of the line from j to i, in seconds. $G_{ij}(0)=1$, so that the signal $e_{ij}(t)$ in the steady state is approximately equal to

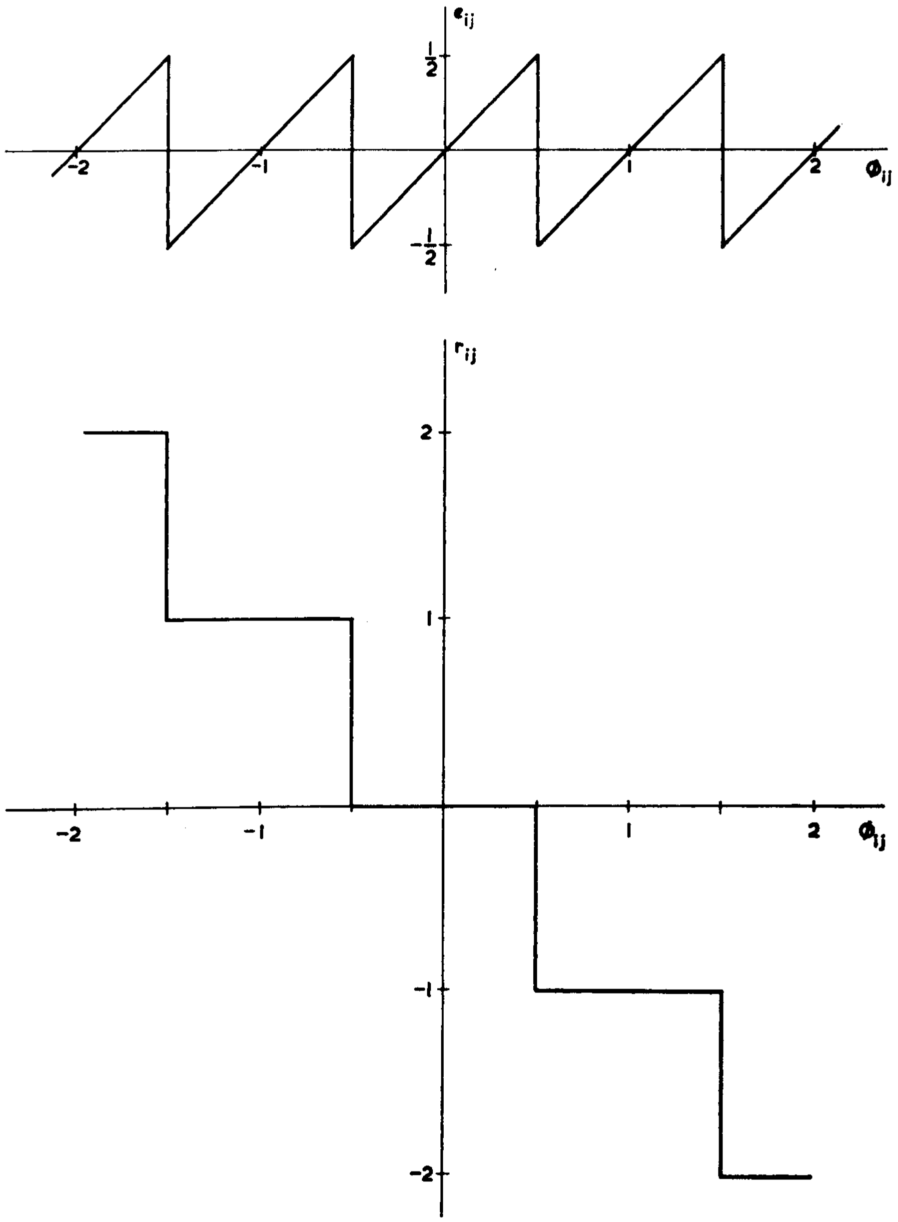

FIG. 1. (a) Phase comparator output characteristic.
(b) Variation of integer r_{ij}.

the true phase difference if, where f_s is the final frequency, d_{ij} is padded out so that $f_s d_{ij}$ is approximately equal to an integral number, N_{ij}, of frames.

A "single-ended" control system

The weighted sum of all the store fill signals available at the input to an exchange is used as a basic error signal. The fill signals are passed through low-pass filters with transfer function $F_{ij}(s)$ and are amplified with a gain k_{ij}. They are then added together and applied to the control input of the oscillator at exchange i, as shown in Fig. 2. Its frequency then becomes

$$f_i(t)=f_{oi}+\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\{e_{ij}(t)\}_{Fij} \tag{4}$$

where f_{oi} is the frequency without control signals applied. Taking Laplace transforms and rewriting in matrix form, we obtain a system equation of the form

$$\{s\mathbf{I}-\mathbf{A}(s)\}\bar{\mathbf{f}}=\mathbf{f}_o+\mathbf{u}(s) \tag{5}$$

where $\bar{\mathbf{f}}$ is the vector of transformed frequencies. The form of matrix $\mathbf{A}(s)$, which contains connection, gain, delay and filter information, determines the stability, while both $\mathbf{f}_o$, the vector of uncontrolled frequencies, and $\mathbf{u}(s)$, a function of various k_{ij}, r_{ij} and d_{ij} terms, determine the final system operating frequency, f_s. A block diagram of this system is shown in Fig. 2. It will be seen that the system is "bilateral", i.e. if $k_{ij}=0$ then $k_{ji}=0$ also; it is not necessary for $k_{ij}=k_{ji}$.

A "double-ended" control system

In this system, shown in Fig. 3, the e_{ij} is transmitted to the distant exchange j where it is compared with the locally generated e_{ji}. The difference is used to form the error signal at exchange j, yielding a different equation for exchange frequency:

$$f_i(t)=f_{oi}+\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\left\{\{e_{ij}(t)\}_{Fij}-\{e_{ji}(t-d_{ij})\}_{Hij}\right\}. \tag{6}$$

A similar matrix equation results:

$$\{s\mathbf{I}-\mathbf{B}(s)\}\ =\mathbf{f}_o+\mathbf{v}(s). \tag{7}$$

The full details of the matrices and vectors in equations (5) and (7) are given in the appendix.

STABILITY

The characteristic equation determining the stability of the single-ended system equations (5) is

$$\det|s\mathbf{I}-\mathbf{A}(s)|=0. \tag{8}$$

If the roots in $s=\sigma+j\omega$ of this equation have positive real parts ($\sigma>0$), then the system is unstable. The Gershgorin Circle theorem [4] is applied to the problem of localising the roots of this determinant when $\sigma>0$, [5, 6]. The special case of this theorem used is sometimes known as the method of "diagonal dominance" [13]. This produces the result that, in the absence of filters, stability is unconditional for all values of k_{ij}. This is similar to the result of GERSHO and KARAFIN [13], quoted by WEST in [14], but here it is not required that

$$\sum_{j=1,\neq i}^{n}(k_{ij})=1\,,$$

as in [13].

The presence of filters will introduce a stability criterion [13] which relates the k_{ij} to the properties of the filters, giving an upper bound on the k_{ij}. For the double-ended system described by equation (7), in the absence of filters, a sufficient stability condition [6, 16], is that

$$(d_{ij}+d_{ji})_{\max\, j,\, j=1,2,\dots n}\left(\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\right)<\sqrt{2}-1\,. \tag{9}$$

Filters will again impose additional restrictions [16]. The above condition is derived by application of the "diagonal dominance" method to the system equations (7). The resulting vector diagram gives inequality (9). The condition of Brilliant, quoted without proof in [7], appears to be derived similarly. He considers the worst case when the e_{ij} and $-e_{ji}$ are added in unequal proportions, namely when $-e_{ji}$ is used alone; this gives a lower value of the upper bound for gain-delay product.

All these conditions can be readily satisfied in practice. The satisfactory operation of a three exchange model of double-ended control [3] was demonstrated by Jarvis early in 1965. The gains used in this model were well within the stability margin and provided adequate control of the oscillator phase differences.

It is interesting to note that for the special case where all delays are equal and all gains are equal, the stability equation for a fully inter-connected network can be factorised to

$$\begin{aligned}&\{s+(n-1)k[1+\exp(-2sd)-2\exp(-sd)]\}\\&\{s+(n-1)k[1+\exp(-2sd)-2\exp(-sd)]\\&\quad+2nk\exp(-sd)\}^{n-1}=0\,.\end{aligned} \tag{10}$$

From the first factor the condition that $(n-1)kd<3\pi/4$ is obtained and from the second factor the condition that $kd<\pi/4$ is obtained. The

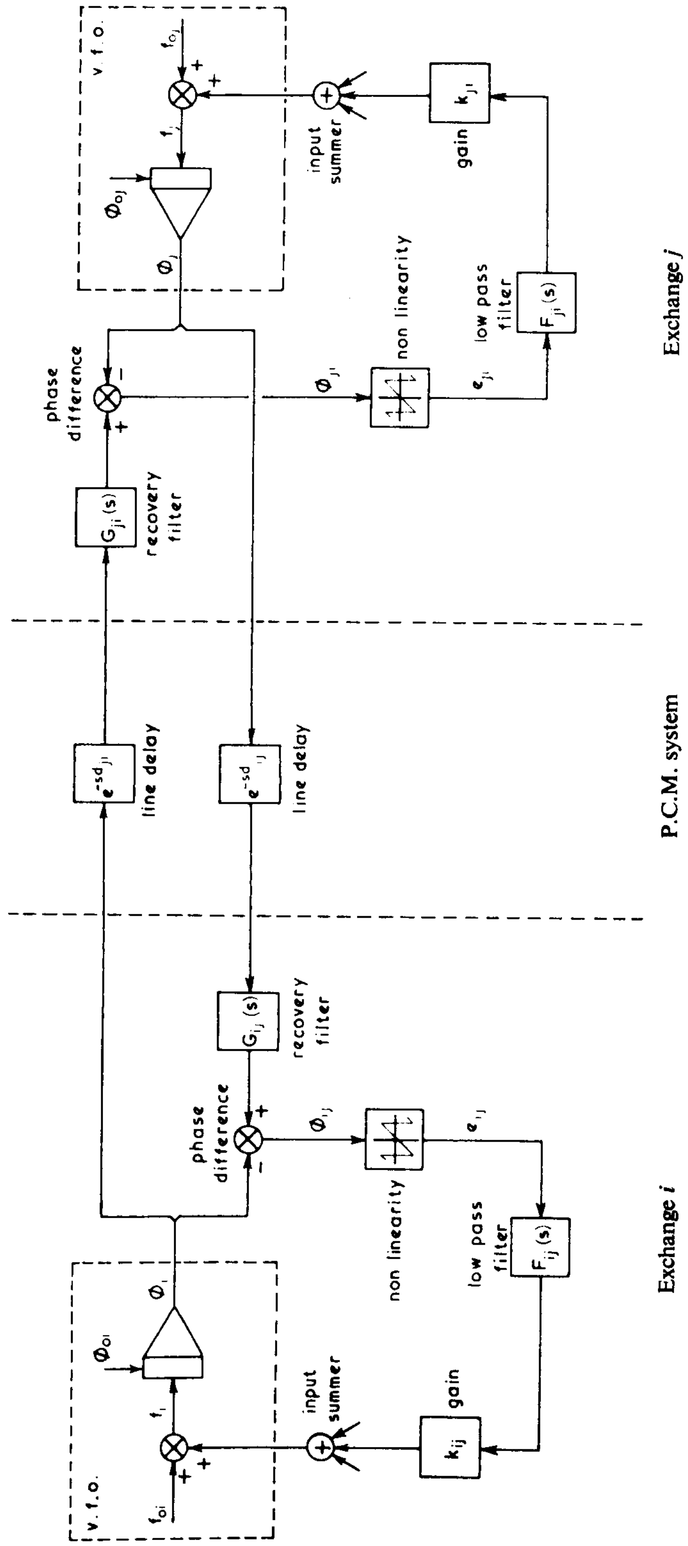

FIG. 2. Single-ended control system.

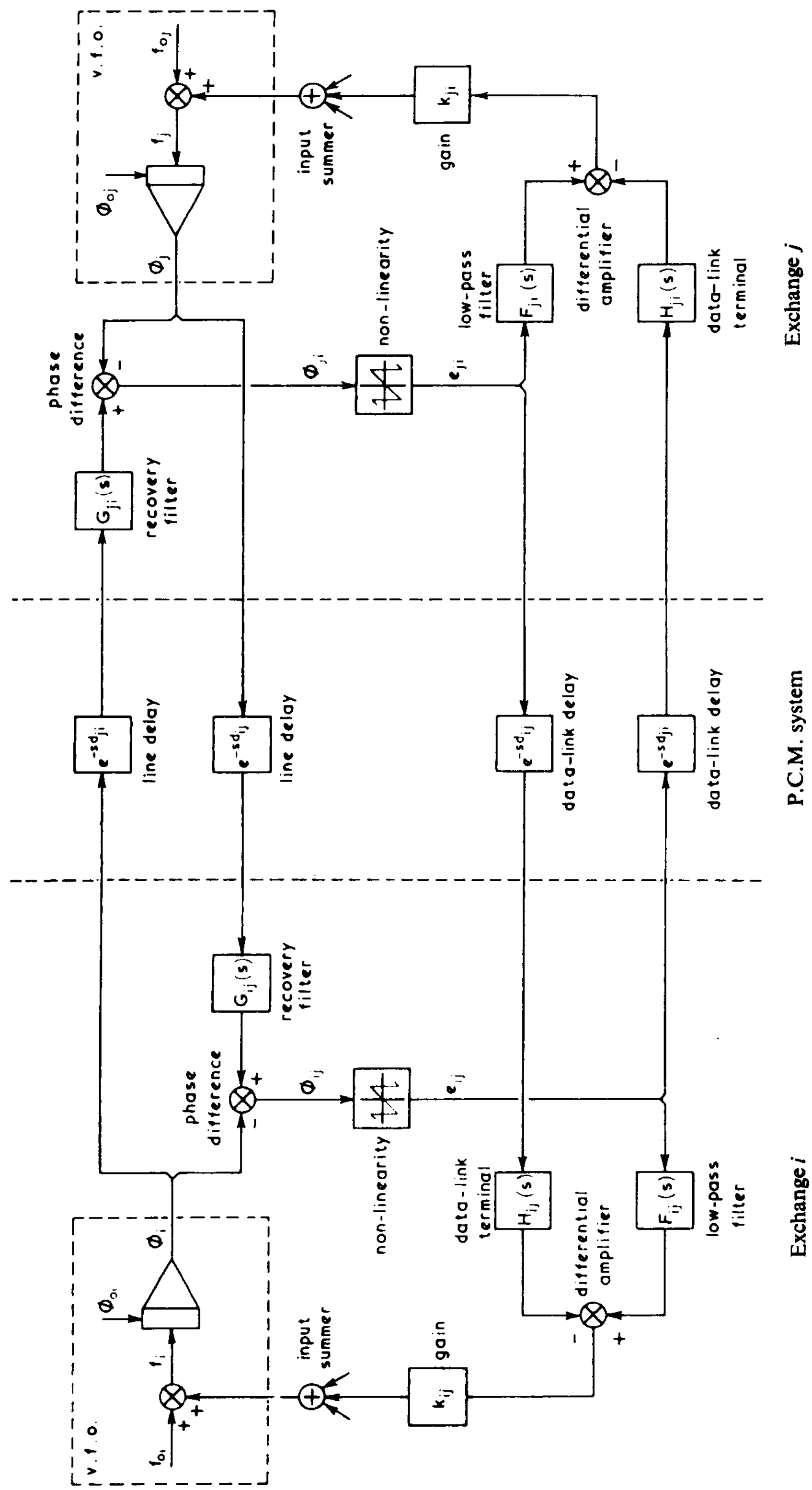

FIG. 3. Double-ended control system.

first is obviously more important for $n > 4$. This is illustrated by Fig. 4, and corrects the result of YAMATO *et al.*, derived in [8].*

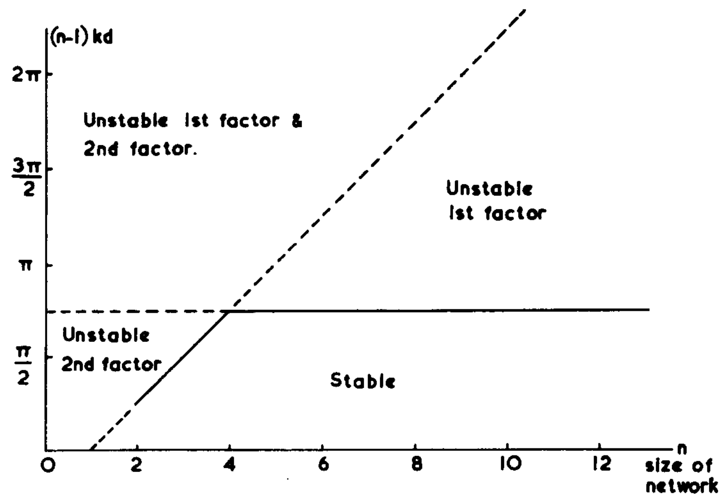

FIG. 4. Stability of fully interconnected double-ended systems.

STEADY STATE CONDITIONS

In the steady state, all the oscillator frequencies will attain a common value, but some residual error signal, dependent on the amplifier gains, is required to drive each frequency from its uncontrolled value to the common steady state value. It will be shown that this value does not depend critically on the phase differences at the instant the control system is first connected, but is largely dependent on the uncontrolled frequencies and the weightings of the error signals.

In the case of the single-ended system, the error signals are the buffer-store "fills" and these are minimised. Since these are functions of the line delays, the system frequency is shown to be a function of these delays. In the case of the double-ended system, the error signals are almost exactly the sums of true phase differences between oscillators and the system frequency can be made independent of the line delays and their variations.

In the correct mode of operation, all the buffer-store fills will be near to zero, as determined by the values of the line delays and the phase differences between oscillators; the actual values will depend on the residual error signals. The store capacities will be such that the stores can bridge the interval between input and output of information.

If a phase comparator passes through a discontinuity in its output characteristic, it is possible for other, larger values of phase difference, and hence store fill, to generate the same apparent phase difference and error signal. The system frequency will also be stable but the store fills will exceed their capacities so that traffic is lost; this will not prevent control signals from passing between the exchanges. This is "wrong mode" operation, explained in more mathematical detail in a later section of this paper.

To discuss the various modes of operation, an $(n-1)$-dimensional state-space is constructed, with $n-1$ principal phase differences as the coordinates. The various comparator discontinuities will define distinct regions in this state-space. It will be found that motion can occur from one region to another. Since the Laplace Transforms in equations (5) and (7) are only valid within a single region, the Final Value Theorem cannot be applied directly to obtain the steady-state conditions. Where this theorem is applied, therefore, it is assumed that the known final region has been entered. It will be seen that the point of entry to a region of the phase-space does not affect the steady-state conditions within that region.

Each region will define a possible steady-state point. This point may or may not lie inside the region defining it. If it does lie in the region, it gives rise to a real operating point and hence a "wrong mode" when the point is not the origin. Points not lying in the region defining them are called virtual operating points. These have no physical reality but are useful when discussing the behaviour of trajectories within the region. All such trajectories will enter other regions where they come under the influence of other operating points. Occasionally, operating points will be encountered that actually lie on the boundary of their defining region.

Provided that the stability criteria, discussed earlier, are satisfied it follows that all real operating points are stable, but operating points lying on the

* The version of Fig. 4 given in the original Symposium paper itself contained an error. The authors wish to thank the referee of this paper who pointed this out.

boundary are quasi-stable. That is, trajectories starting within the region and in the neighbourhood of this point will approach it as time increases, but any small disturbance can start a trajectory into an adjacent region.

In a large network there may be a great number of real stable operating points. It is the existence of these points which gives rise to the problem of wrong mode operation.

SYSTEM OPERATING FREQUENCY

Provided that stability has been established, the Laplace Final Value theorem may then be applied to the system equations, as above, to yield an expression for the final frequency. This is the ratio of two determinants and putting, for all i and j, $k_{ij}=a_{ij}.k_i$, the ratio can be considerably simplified. The $a_{ij}=a_{ji}$ and will take the values of 1 or 0 as the ith and jth stations are, or are not, respectively, directly connected together. This has been done by other authors, such as BRILLIANT [12], to obtain an algebraic expression, but it is possible to simplify these still further without further restrictions on the parameters [16]. We take the filter transfer functions as:

$$F_{ij}(s)=F_{ij}(0)-s.T_{Fij}+\text{higher powers of } s.$$

T_{Gij}, T_{Hij} are similarly defined. It is assumed further that all $F_{ij}(0)=F$, with G and H similarly defined. With these assumptions for the values of $k_{ij}.F_{ij}(0)$, etc., the expression for the final frequency of the double-ended system may be reduced to:

$$f_s=\frac{\sum_{i=1}^{n}(f_{oi}/k_i)+(F-H)\sum_{i=1}^{n}\sum_{j=1}^{n}r_{ij}+(F-H)(G-1)\sum_{i=1}^{n}\sum_{j=1,\neq i}^{n}a_{ij}.\phi_{oi}}{\sum_{i=1}^{n}(1/k_i)+\sum_{i=1}^{n}\sum_{j=1}^{n}\{(G-1)(T_{Fij}-T_{Hij}-H.d_{ij})+(F-H)(T_{Gij}+G.d_{ij})\}}.$$

The system is such that $G=1$; for the single-ended system the same formula can be derived but with $H=0$. The system frequency of the single-ended system is thus:

$$f_s=\frac{\sum_{i=1}^{n}(f_{oi}/k_i)+F\sum_{i=1}^{n}\sum_{j=1}^{n}r_{ij}}{\sum_{i=1}^{n}(1/k_i)+F\sum_{i=1}^{n}\sum_{j=1}^{n}(T_{Gij}+d_{ij})}. \qquad (11)$$

This only applies when the r_{ij} are constant throughout the transient motion, as discussed above.

For the double-ended system:

$$f_s=\frac{\sum_{i=1}^{n}(f_{oi}/k_i)+(F-H)\sum_{i=1}^{n}\sum_{j=1}^{n}r_{ij}}{\sum_{i=1}^{n}(1/k_i)+(F-H)\sum_{i=1}^{n}\sum_{j=1}^{n}(T_{Gij}+d_{ij})}. \qquad (12)$$

Usually $F=H=1$ in which case (12) reduces to:

$$f_s=\frac{\sum_{i=1}^{n}(f_{oi}/k_i)}{\sum_{i=1}^{n}(1/k_i)} \qquad (13)$$

which is independent of delays or operating mode; i.e. the value of f_s is the same in all regions of the phase-space. This feature of the double-ended system is of great importance and makes it preferable to the single-ended system.

SYSTEM OPERATING MODE

The operating mode is not unique [9, 10] and quite large phase differences can exist between exchanges operating at a common frequency. This is due to the many possible values of the variables $r_{ij}(t)$ in the system equations as the phase detector discontinuities are traversed. New variables $q_{ij}(t)$, N_{ij} and δ_{ij} are defined for substitution into equations (1) and (3). The integer N_{ij} and the incremental delay δ_{ij} are defined by:

$$\int_{t-d_{ij}}^{t}\{f_j(\tau)\}_{Gij}.\mathrm{d}\tau=N_{ij}+\int_{t-\delta_{ij}}^{t}\{f_j(\tau)\}_{Gij}.\mathrm{d}\tau,$$

where

$$-\tfrac{1}{2}<\int_{t-\delta_{ij}}^{t}\{f_j(\tau)\}_{Gij}.d\tau<+\tfrac{1}{2},$$

and the integer $q_{ij}(t)$ by:

$$q_{ij}(t)=r_{ij}(t)+N_{ij}.$$

In the steady state these relations become:

$$f_s.d_{ij}=N_{ij}+f_s.\delta_{ij} \quad \text{and} \quad q_{ij}=r_{ij}+N_{ij}$$

where

$$-\tfrac{1}{2}<f_s.\delta_{ij}<+\tfrac{1}{2}.$$

In the close vicinity of the steady state point the q_{ij} and r_{ij} can usually be considered as constants. If $\delta_{ij}=0$ and $\phi_i=\phi_j$, then $q_{ij}=0$.

Putting $k_{ij}=k.a_{ij}$ for all i, j, where $a_{ij}=a_{ji}=1$ or 0, defining

$$a_{ii}=-\sum_{j=1,\neq i}^{n}a_{ij}$$

and assuming that $|\phi_i-f_s|\ll f_s$, all i, the system equations for the single-ended system may be written as:

$$\begin{aligned}(\mathbf{I}+k.\mathbf{D})(\boldsymbol{\phi}-f_s.\mathbf{c}) &= k.\mathbf{A}.\boldsymbol{\phi}+\mathbf{f}_o-f_s.\mathbf{c}\\ &\quad -k(\mathbf{R}+f_s.\mathbf{D})\mathbf{c}\\ &= k.\mathbf{A}.\boldsymbol{\phi}+(\mathbf{f}_o-f_s.\mathbf{c})\\ &\quad -k(\mathbf{Q}+f_s.\boldsymbol{\Delta})\mathbf{c}\\ &= \mathbf{0} \quad \text{in the steady state.} \qquad (14)\end{aligned}$$

$\mathbf{D}$ is the matrix of d_{ij}, $\mathbf{A}$ is the matrix of a_{ij}, $\mathbf{R}$ of r_{ij}, $\mathbf{Q}$ of q_{ij}, $\boldsymbol{\Delta}$ of δ_{ij}, and $\mathbf{c}'=\{1, 1, 1, \dots 1\}$. Where $a_{ij}=a_{ji}=0$, $d_{ij}=\delta_{ij}=r_{ij}=q_{ij}=0$ also.

The general phase differences, as measured by the comparators, are expressed in terms of the $n-1$ phase differences $\phi_1-\phi_n$, $\phi_2-\phi_n$, $\dots$ $\phi_{n-1}-\phi_n$; these are written as the $n-1$ column vector $\mathbf{x}$. In matrix form, $\mathbf{x}=\mathbf{B}.\boldsymbol{\phi}$, where

$$\mathbf{B}=\begin{bmatrix} 1 & 0 & 0 & . & . & 0 & -1\\ 0 & 1 & 0 & . & . & 0 & -1\\ 0 & 0 & 1 & . & . & 0 & -1\\ . & . & . & . & . & . & .\\ 0 & 0 & 0 & . & . & 1 & -1\end{bmatrix}$$

and $\mathbf{B}.\mathbf{c}=\mathbf{0}$.

After performing this subtraction, the resulting $n-1$ equations may be written in matrix form; the new matrices $\mathbf{A}^*$ and $\mathbf{D}^*$, defined by the subtraction, will satisfy $\mathbf{A}^*.\mathbf{B}=\mathbf{B}.\mathbf{A}$, $\mathbf{D}^*.\mathbf{B}=\mathbf{B}.\mathbf{D}$. The equations (14) now become:

$$\begin{aligned}[\mathbf{I}+k.\mathbf{D}^*]\dot{\mathbf{x}} &= k.\mathbf{A}^*.\mathbf{x}+\mathbf{B}.\mathbf{f}_o-k.\mathbf{B}[\mathbf{Q}+f_s.\boldsymbol{\Delta}]\mathbf{c}\\ &= k.\mathbf{A}^*.\mathbf{x}+\mathbf{B}.\mathbf{f}_o-kf_s.\mathbf{B}.\boldsymbol{\Delta}.\mathbf{c}-k.\boldsymbol{\lambda}\end{aligned} \qquad (15)$$

where the $n-1$ column vector $\boldsymbol{\lambda}=\mathbf{B}.\mathbf{Q}.\mathbf{c}$ determines the operating mode.

In the steady state, $\dot{\mathbf{x}}=\mathbf{0}$ so the phase differences $\mathbf{x}$ will depend upon the differences between uncontrolled frequencies, variations of line delay from multiples of one "frame", and upon $\boldsymbol{\lambda}$. The first two effects will usually be small, but the possible non-zero values of $\boldsymbol{\lambda}$ can have considerable effect. Neglecting the frequency and delay effects, non-zero solutions in $\mathbf{x}$ are sought, such that the sum at each exchange of the phase differences as measured by the phase comparators is zero; this is precisely wrong mode operation as $\dot{\mathbf{x}}=\mathbf{0}$. $\boldsymbol{\lambda}=\mathbf{0}$, however, corresponds to the correct mode of operation, desired in practice as the store fills are minimised.

The $\boldsymbol{\lambda}$ will have integer entries, positive, negative or zero. In general

$$\lambda_i=\sum_{j=1}^{n} q_{ij}-\sum_{j=1}^{n} q_{nj} \quad \text{where } q_{ij}=[\tfrac{1}{2}+x_i-x_j-f_s\delta_{ij}]$$

and the matrix $\mathbf{B}$ is of the form shown. Increasing any x_i in steps of 1 will cause the corresponding q_{ij} to increase in steps of 1 and the q_{ji} to decrease in steps of 1. By translation of the origin, it may be seen that the phase space is divided into hypercubes all identical to the region $|x_i|<\frac{1}{2}$ for all $i=1, 2, \dots (n-1)$. It is thus possible to consider only this region and, by putting $\boldsymbol{\Delta}=\mathbf{0}$, to obtain:

$$\lambda_i=\sum_{j=1}^{n-1} q_{ij} \qquad \text{for which } \sum_{i=1}^{n-1} \lambda_i=0.$$

This basic hypercube of the state space is intersected by hyperplanes each corresponding to the overflow of a phase comparator. The hyperplanes define regions in which all the q_{ij} are constant and thereby generate a value of $\boldsymbol{\lambda}$ peculiar to the whole region. This vector $\boldsymbol{\lambda}$ defines a point $\mathbf{x}=\mathbf{A}^{*-1}.\boldsymbol{\lambda}$; this point may lie inside the region generating the $\boldsymbol{\lambda}$ in which case it defines an operating mode.

Trajectories of $\mathbf{x}$ starting within the region move towards this point and will tend to it as $t\to\infty$ unless a bounding hyperplane is first encountered. If the point $\mathbf{A}^{*-1}.\boldsymbol{\lambda}$ does not lie in the region generating this $\boldsymbol{\lambda}$, the trajectories will leave the region, at which stage a different value of $\boldsymbol{\lambda}$ will apply.

The trajectories will change direction and approach the newly defined point. Curved trajectories, in a region defining a wrong mode or containing the origin, may leave the region and arrive at a different steady-state point. Thus the region of attraction of an operating point is not necessarily identical to the region defining it.

Another possibility [9] is that no $\boldsymbol{\lambda}$, including $\mathbf{0}$, will generate a point lying in its corresponding region. This could occur particularly with large differences in uncontrolled frequency or with values of $|f_s.\delta_{ij}|\to\frac{1}{2}$.

The general problem of finding all possible modes for a general network is unsolved, but for a particular case certain modes may readily be found. Moreover, it is impossible for such modes to occur in the region $|x_i|<\frac{1}{2}$, as no overflows can then occur. This property is made use of later.

Certain cases, such as bilateral systems with $n=4$, have some or all of the equilibrium points, other than the origin of $\mathbf{x}$, actually on the inter-regional boundaries. Thus the trajectories from one region enter the "equilibrium" point, at which stage a new value of $\boldsymbol{\lambda}$ is generated and the motion proceeds into the next region.

The fully interconnected case is particularly easy to discuss, since the matrix $\mathbf{A}^*=n.\mathbf{I}$ and the trajectories are therefore always straight lines to the point for which $\dot{\mathbf{x}}=\mathbf{0}$. For this case, in the steady state:

$$\mathbf{x}=(1/n)\boldsymbol{\lambda}+(fs/n)\mathbf{B}.\boldsymbol{\Delta}.\mathbf{c}-(1/nk)\mathbf{B}.\mathbf{f}_o \qquad (16)$$

Substituting for q_{ij}:

$$\lambda_i = \sum_{j=1}^{n-1} [x_i - x_j - f_s\delta_{ij} + \tfrac{1}{2}]$$
$$= \sum_{j=1}^{n-1} \left[(\lambda_i - \lambda_j)/n - f_s\left\{\delta_{ij} - \sum_{l=1}^{n} (\delta_{il} - \delta_{jl})/n\right\} + (f_{oj} - f_{oi})/nk + \tfrac{1}{2} \right]. \quad (17)$$

This equation must be satisfied by all vectors $\boldsymbol{\lambda}$ which give a stable operating mode. When $n=3$ the possible values of $\boldsymbol{\lambda}$ are $(^{-1}_{1})$ and $(_{-1}^{1})$, and the corresponding phase states are surrounded by regions in which these values of $\boldsymbol{\lambda}$ are generated. Thus a sufficiently small variation of either $\mathbf{f}_0$ or of $\boldsymbol{\Delta}$ will still lead to stable states in these regions. As $\boldsymbol{\Delta}$ is varied, additional regions of constant $\boldsymbol{\lambda}$ are produced and these can be shown to contain no stable operating point in the case of $n=3$. Figures 5 and 6 indicate the effect of varying $\mathbf{f}_0$ and $\boldsymbol{\Delta}$ separately. It will be noted that equation (17) must be modified by the addition of extra terms when $\boldsymbol{\Delta}$ is such that q_{in} is non-zero for any $i=1, 2, \ldots (n-1)$.

The case of $n=4$ is somewhat complicated by the fact that the function r_{ij} is not single valued at the transition point. Thus although at first sight the vectors corresponding to the possible equilibrium points do not satisfy equation (17), they are in fact quasi-stable points since they lie on the boundary of the region. The cube of phase-space defined by $|x_1| < \frac{1}{2}$ is divided into thirteen pieces, one of which contains the origin and the points $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, six pieces containing the other corners and six pentahedra. An example of each is shown in Fig. 7 together with the trajectories for the piece. To show the inter-relation between the various quasi-stable points and the origin-region, the cube is projected onto a plane normal to the line (111), as in Fig. 8. Certain of the regions do in fact have trajectories into the next cube, which is similar to this in all respects. The lines are thus shown as re-entering the cube on the opposite face.

The only stable point is the origin (0, 0, 0); the other points for which $\dot{\mathbf{x}}$ should be zero are the six permutations of $(\frac{1}{4}, -\frac{1}{4}, 0)$ and the six permutations of $(\pm\frac{1}{4}, \mp\frac{1}{2}, \pm\frac{1}{4})$.

It may be shown that for higher order fully interconnected systems, where n is divisible by 4, many of the possible modes are quasi-stable points, but other modes exist which are perfectly stable. For the case of $n=4$ but without full interconnection, the number of regions is reduced by removal of the boundaries due to the phase-detector discontinuities, but again ~~all the~~ some possible mode points are quasi-stable.

TRANSIENTS

Some knowledge of the transient motion is necessary in order to define the range of initial phase differences from which the system will relax to a steady state within the origin-region for correct mode operation. In addition it is necessary to

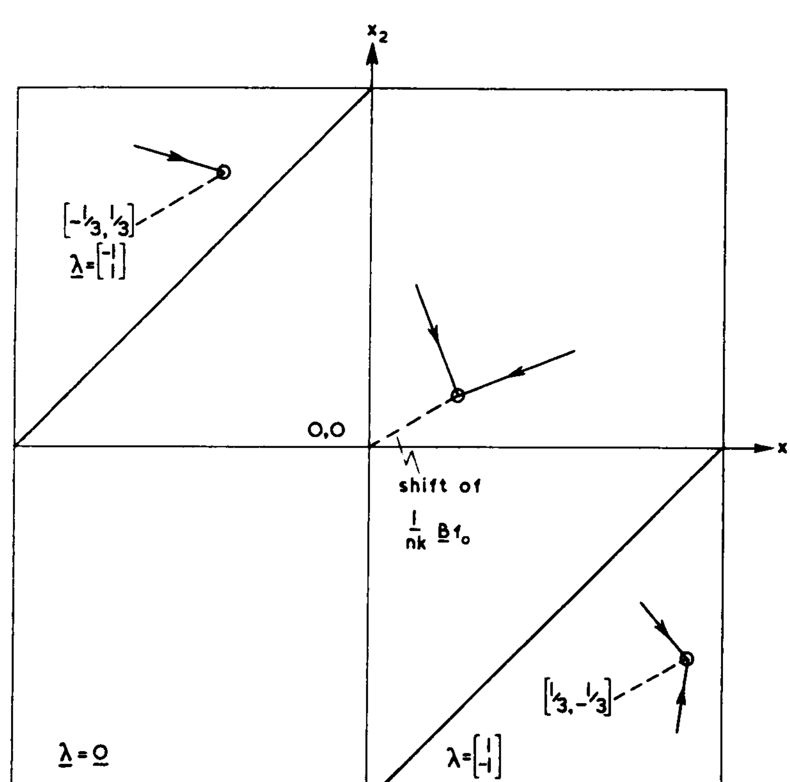

FIG. 5. Variation of f.

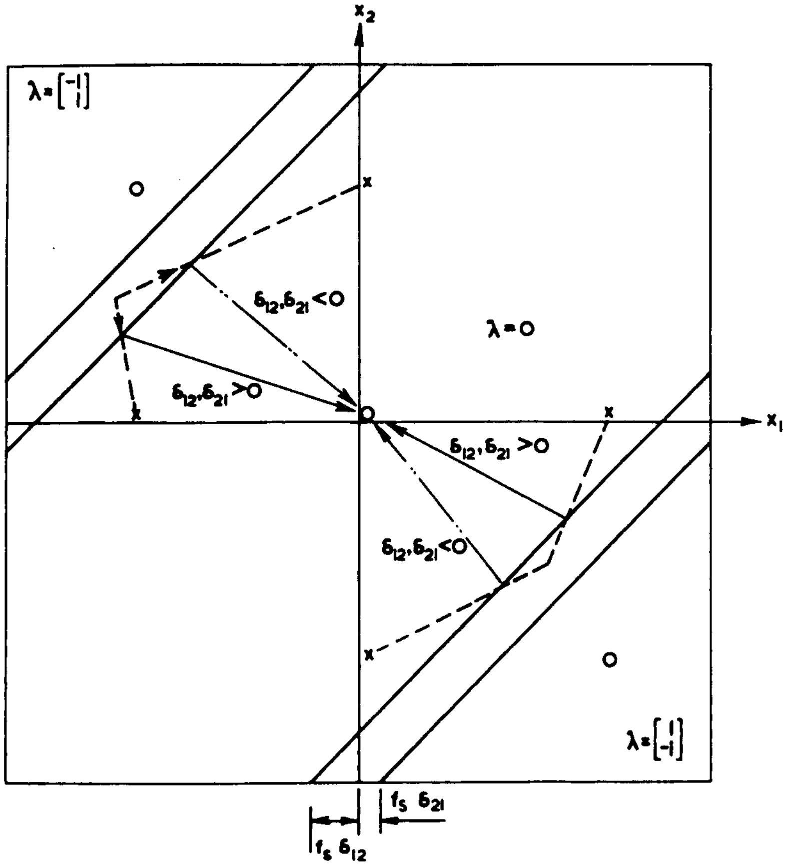

FIG. 6. Effect of adding small delays while $\mathbf{B}.\mathbf{f}_0=0$.
o=real operating point
x=virtual operating point
All $\delta_{ij}=0$ except δ_{12}, δ_{21}. Note shift of operating points by $(f_s/n).\mathbf{B}.\Delta.\mathbf{c}=\frac{1}{2}\begin{pmatrix}\delta_{12}\\ \delta_{21}\end{pmatrix}f_s$.

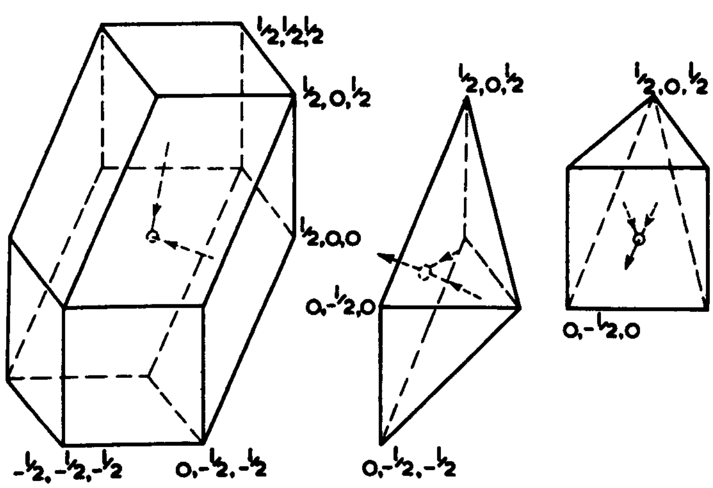

FIG. 7. Sections of cube.

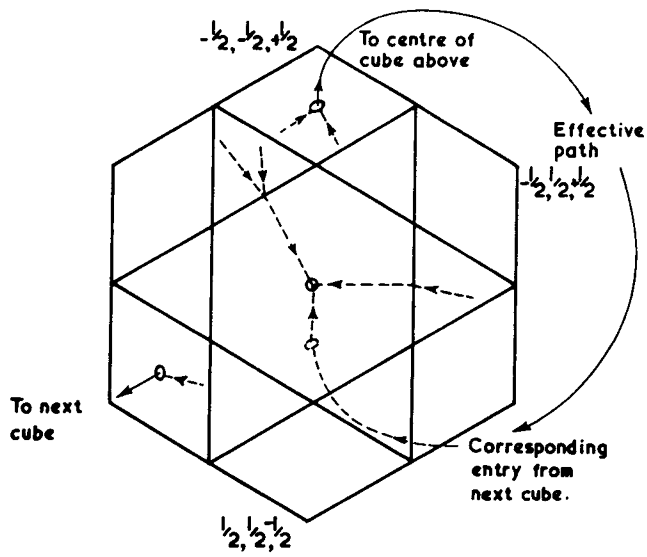

FIG. 8. Projection of cube onto plane $x_1+x_2+x_3=0$.

determine the maximum size of disturbance which the system can tolerate before the correct mode is left; this is proportional to that which can be tolerated before the stores overflow. For the fully inter-connected network without delays the transient motion is monotonic and all trajectories are straight lines into the origin. For less connected networks the trajectories will not be straight lines and may well leave the zone of the correct mode. Consideration of a Liapunov function derived from the phase difference equations (15) [11] shows that the Euclidean state-space norm $\rho(t)$ defined by $\rho^2(t)=\mathbf{x}'(t).\mathbf{x}(t)$ is never greater than its initial value at $t=0$. Thus $\mathbf{x}(t)$ lies within the hypersphere whose centre is the origin and whose radius is $\rho(0)$. Its speed of return to the origin will depend on the degree of interconnection and on the gain of the amplifiers; highly interconnected networks are preferable to thinly connected networks. If the radius of the hypersphere is less than $\frac{1}{4}$ then the network cannot enter a region in which wrong mode operation is possible.

In the event of a failure of a part of the control system or an inter-exchange link, the system will be designed in such a way as to reduce the degree of interconnection of the network. This will give rise to transients in addition to any generated by the fault itself before it is disconnected. The re-introduction of equipment after repair is in fact the same operation as initial set-up of that part of the system and will also produce transients. There may be some case for the inclusion of limiters in parts of the control system to reduce the frequency shifts caused by the insertion of links between stations whose phases differ by any significant amount. The gains will have to be such that the transients due to these operations are adequately and rapidly dispersed.

DETECTION OF INCORRECT MODES

Should the network enter an undesirable operation mode, means must be available to detect this condition. It can be shown that for any network configuration at least one of the $|x_i|>\frac{1}{4}$ for such a mode to govern the system operation. Thus provided that the detection equipment knows the true phase of every station, it can give an alarm if this phase is outside the limit for more than a short time. In a typical network, every exchange is connected in such a way that one important exchange can be reached over not more than four links. If this important exchange be treated as having zero phase, then for the other exchange to have a phase of more than $\frac{1}{4}$, the phase difference across at least one link must exceed $\frac{1}{16}$ of a frame, which is rather more than the expected variation due to delay changes affecting the phase detectors. Additionally, the phase differences can be signalled to small computing centres within the network in such a way as to allow comparison of the true phases with the limits of $\pm\frac{1}{4}$. Failure of the computing centres need not affect the normal mode operation of the system as no alarms will be generated in this mode. When an alarm is produced, the action required is for every exchange to synchronise to its parent alone, which creates a network with only one possible operating mode. As soon as the alarm condition is removed the network will revert to the more secure highly connected form. Correct adjustment of system gains will ensure that no transient caused by this process could cause another wrong mode to be entered.

This method of correction [15] does not rely on detailed knowledge of the mode patterns which are possible for a particular network configuration. The number of such patterns will in general be large and the network configuration may well change due to a fault, the transient from which could cause the wrong mode. For practical purposes, therefore, the existing knowledge of operating modes should suffice.

INITIAL SETTING OF DELAYS

It has been stated that the line delays are to be padded-out to equal an integral number of frame periods. This raises the question of how this is to be done and it may seem at first sight that some absolute measurement of line delay is necessary. Such a measurement would be very difficult if not impossible to make.

Fortunately in a network of n exchanges there is some arbitrariness in defining phases and phase differences. Phase differences for the whole network are uniquely defined once a given set of $n-1$ phase differences, involving all the n exchanges, is known. In discussion of operating mode we have used such a set, namely $\phi_i - \phi_n$ $(i = 1, 2, \ldots n-1)$.

This fact has a direct bearing on the padding-out problem. Theoretically all that is required is that, for each of the $n-1$ links across which one of the $n-1$ basic phase differences are measured, $d_{ij}+d_{ji}$ is an integral number of frame periods. This is easily carried out by measurement and adjustment on a loop basis between pairs of exchanges. Once this has been done all other unidirectional delays d_{ij} are uniquely defined and must be adjusted individually by suitable measurement around a closed path consisting of the unidirectional half-link from j to i and other half-links that have already been adjusted.

The arbitrariness has no real effect on the system operation: the state-space diagram of hypercubes and operating points is translated in its entirety, relative to the true phase-difference axes, by an unknown quantity. This quantity would only be known if the unidirectional delays in the $n-1$ basic links could be measured.

The buffer stores, of course, will all be within their operating range in the correct mode, as they are set to the middle of their range during the padding-out process.

SAITO [9] states that $n-1$ delays are arbitrary but does not specify that they must be chosen to link all n exchanges.

CONCLUSIONS

This paper has studied a number of problems of p.c.m. oscillator control systems. These systems are, mathematically, piece-wise linear multivariable control systems with time delays, in which the precise connections and hence equations are not always known in advance. Despite these difficulties much useful knowledge has been obtained concerning final frequencies, stability and operating modes. These modes are a peculiar feature of these systems, brought about by the piece-wise periodic error detectors in the network. Some of the results given are new, other results are generalisations of earlier work by the present authors as well as others working in the field.

More work on the dynamic behaviour of these systems is required. The problem of finding all possible modes of a given system has not yet been solved. However, sufficient knowledge has been gained to allow a field trial to be planned. This is now being undertaken by the British Post Office.

In designing a particular network, a careful choice of the gains k_{ij} has to be made. Using ideas developed in this paper, the following procedure for choosing the k_{ij} is recommended:

(i) From a study of the $n-1$ dimensional phase difference diagram, in the case of this network but with zero line delays, the k_{ij} must be chosen large enough for an equilibrium frequency to exist, given the likely magnitudes of the components of the uncontrolled frequency difference vector $\mathbf{B}.\mathbf{f}_0$. In Fig. 5 this corresponds to a choice of k sufficiently large to keep the shift $(1/nk)\mathbf{B}.\mathbf{f}_0$ in the $\boldsymbol{\lambda}=\mathbf{0}$ operating point within the hexagonal $\boldsymbol{\lambda}=\mathbf{0}$ region.

(ii) The $n-1$ dimensional diagram and the lower bound on the k_{ij} will have to be modified slightly to account for time delays as in Fig. 6. The $f_s.\delta_{ij}$ effect on the diagram is small as a result of the padding out process.

(iii) An upper bound on the k_{ij} must be calculated from stability considerations. The sufficient conditions in this paper will be adequate at least for initial design purposes. Once stability is assured, f_s can be calculated from the appropriate formula given in the paper; f_s must lie within the controllable range of each oscillator.

(iv) An essential feature of the design process is to keep the necessary buffer store size as small as possible. A certain size is required to accommodate the small and slowly changing $f_s.\delta_{ij}$. Further store capacity will be required to accommodate the steady state phase differences; these will come about because of the scatter of the uncontrolled frequencies $\mathbf{f}_0$ and are reduced by an increase in the k_{ij}. This consideration produces a second lower bound on the k_{ij} likely to be more critical than the bound in (i) above.

The study of transients shows that the ideal instant for connection of the control system is when all the phase differences are near zero; in practice, the network could be built up progressively, exchanges being added to the control when their phases came close to that of their connected neighbours. This process would probably be automated so that faults could be cleared quickly using the same procedure.

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APPENDIX: LAPLACE TRANSFORMS

We have to obtain the Laplace transforms of equations (4)

$$f_i(t)=f_{oi}+\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\,.\,\{e_{ij}(t)\}_{Fij}$$

and (6)

$$f_i(t)=f_{oi}+\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\,.\,[\{e_{ij}(t)\}_{Fij}-\{e_{ji}(t-d_{ij})\}_{Hij}]$$

where, from equations (1) and (3),

$$e_{ij}(t)=\{\phi_j(t)\}_{Gij}-\int_{t-d_{ij}}^{t}\{f_j(\tau)\}_{Gij}\,.\,d\tau-\phi_i(t)+r_{ij}(t).$$

The variable $r_{ij}(t)$, defined by equation (2), may be taken as a constant r_{ij} throughout the linear range of the phase comparator. With this restriction, we obtain the transform of equation (4) and write it in the matrix form (5)

$$[s\mathbf{I}-\mathbf{A}(s)]\bar{\mathbf{f}}=\mathbf{f}_o+\mathbf{u}(s)$$

where $\bar{\mathbf{f}}$ is the vector of transformed frequencies, $\mathbf{f}_o$ is the vector of the uncontrolled frequencies and the matrix $\mathbf{A}(s)$ and vector $\mathbf{u}(s)$ are defined by:

$$a_{ii}(s)=-\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\,.\,F_{ij}(s)$$

$$a_{ij}(s)=+k_{ij}\,.\,F_{ij}(s)\,.\,G_{ij}(s)\,.\,\exp(-sd_{ij})\ \text{for}\ i\neq j$$

and

$$u_i(s)=\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\,.\,F_{ij}(s)\,.\left\{r_{ij}+G_{ij}(s)\,.\,\phi_{oj}-\phi_{oi}\right.$$
$$\left.-G_{ij}(s)\left(\frac{f_{oj}}{s}\right)(1-sd_{ij}-e^{-sd_{ij}})\right\}.$$

From equation (6) we similarly obtain equation (7)

$$[s\mathbf{I}-\mathbf{B}(s)]\bar{\mathbf{f}}=\mathbf{f}_o+\mathbf{v}(s)$$

for which

$$b_{ii}(s)=-\sum_{\substack{j\neq 1\\ \neq i}}^{n} k_{ij}\,.\,[F_{ij}(s)$$
$$+G_{ij}(s)\,.\,H_{ij}(s)\,.\,\exp(-sd_{ij}-sd_{ji})]$$

$$b_{ij}(s)=+k_{ij}\,.\,\{F_{ij}(s)\,.\,G_{ij}(s)+H_{ij}(s)\}\exp(-sd_{ij})$$

and

$$v_i(s)=\sum_{\substack{j=1\\ \neq i}}^{n} k_{ij}\,.\left[F_{ij}(s)\,.\,r_{ij}-H_{ij}(s)\,.\,r_{ji}\right.$$
$$+\{F_{ij}(s)\,.\,G_{ij}(s)+H_{ij}(s)\}\,.\left\{\phi_{oj}\right.$$
$$\left.+\frac{f_{oj}}{s}[1-sd_{ij}-\exp(-sd_{ij})]\right\}$$
$$-\{F_{ij}(s)+G_{ij}(s)\,.\,H_{ij}(s)\}\,.\left\{\phi_{oi}\right.$$
$$\left.\left.+\frac{{}^{oi}}{s}[(1-sd_{ij}-sd_{ji}-\exp(-sd_{ij}-sd_{ji})]\right\}\right].$$

In deriving the system operating frequency, it is to be observed that all the terms

$$(f_{oj}/s)[1-sd_{ij}-\exp(-sd_{ij})]$$

have zero value at $s=0$. The ϕ_{oj} terms which remain will be seen to cancel as $G_{ij}(0)=1$.

The initial conditions for these transforms are that at $t=0$, $\phi_i(t)=\phi_{oi}$ etc. For $t<0$, $\phi_i(t)=\phi_{oi}+f_{oi}\,.\,t$, since $f_i(t)=f_{oi}$ for all $t<0$.

Résumé—Pour un fonctionnement synchronisé des centraux téléphoniques à modulation de codes d'impulsions il est souhaitable d'assurer l'égalité de frequence des oscillateurs à chaque central. Un système de commande à variables multiples employant une technique à phase verrouillèe peut être utilisé pour produire une synchronisation à long terme.

Beaucoup d'auteurs ont été actifs dans ce domaine et le présent article indique certains des résultats d'analyse de la stabilité et de la fréquence de fonctionnement, avec une reférence particulière aux travaux de ces auteurs. Un système important, connu sous le nom de "système à deux extremités", n'a pas été traité par la plupart des auteurs antérieurs, quoiqu'il soit susceptible de constituer la base d'un système pratique. Il est décrit ici et des détails de son fonctionnement sont donnés.

Le critère de performance du système consiste à avoir pour deux centraux connectés quelconques, une différence de phase ne depassant pas une certaine valeur fixe. Certains modes de fonctionnement du système violent cette condition et ils sont examinés. Des moyens pour les éliminer sont suggérés. Le comportement transitoire peut être examiné en employant des fonctions de Liapunov.

Zusammenfassung—Beim synchronisierten Betrieb von PCM-Fernsprechämtern ist in jedem Amt die Gleichheit der Oszillatorfrequenz zu sichern. Um eine Langzeitsynchronisation zu erzielen, kann ein mehrvariables Regelungssystem aufgebaut werden, in dem eine phasenatarre Technik angewendet wird.

Viele Autoren waren schon auf diesem Gebiet tätig und so werden in dieser Arbeit einige Ergebnisse der Analyse der Stabilität und der Betriebsfrequenz mit besonderem Bezug auf die Arbeit dieser Autoren angedeutet. Ein wichtiges System, das als "doppelendiges System" bekannt ist, wurde von den meisten früheren Autoren nicht behandelt. Es wird hier beschrieben und es werden Einzelheiten seines Verhaltens angegeben.

Das Kriterium für das Verhalten des Systems liegt darin, daß zwei miteinander verbundene Ämter eine unter einem festen Betrag liegende Phasendifferenz haben müssen. Bestimmte Betriebsarten des Systems verletzen diese Bedingung und werden untersucht. Der Übergangsprozeß kann durch Anwendung von Ljapunow-Funktionen geprüft werden.

Резюме—Для синхронизированной работы импульсно-модулируемых телефонных станций желательно обеспечить равенство частот импульсных генераторов на каждой станции. Многокоординатная система управления с техникой зафиксированной фазы может быть использована для осуществления долгосрочной синхронизации.

Много авторов работали в этой области и настоящая статья указивает некоторые из результатов анализа устойчивости и используемой частоты, с особым вниманием к трудам ее авторов. Важная система, известная под именем "двухконечной системы", не рассматривалась большинством из предыдущих авторов, хотя она способна составить основу практической системы. Она здесь описывается и даются детали ее работы.

Критерий работы системы состоит в том что две любые соединенные станции должны иметь разницу фаз не превышающую некоторую заданную величину. Некоторые методы работы системы нарушают это условие и они рассматриваются. Предлагаются способы их устранения. Переходное поведение может быть рассмотрено при помощи функций Ляпунова.

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